GAUSSIAN DISTRIBUTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.3.

Probably the most common continuous probability density is the gaussian distribution, specified by

$$\rho(x) = Ae^{-\lambda(x-a)^2} \tag{0.1}$$

First, we need to normalize the distribution by finding A. That is, we must have

$$A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = 1 \tag{0.2}$$

The gaussian integral is very common, and the result is that

$$\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = \sqrt{\frac{\pi}{\lambda}} \tag{0.3}$$

$$A = \sqrt{\frac{\lambda}{\pi}} \tag{0.4}$$

Although there is no closed form indefinite integral, the definite integral can be found by a cute trick.

$$\left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \tag{0.5}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy \tag{0.6}$$

We can now transform to polar coordinates using

$$r^2 = x^2+y^2 \tag{0.7}$$
$$dx\,dy = r\,dr\,d\theta \tag{0.8}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy = \int_0^{\infty}\int_0^{2\pi} re^{-r^2}d\theta\,dr \tag{0.9}$$
$$= 2\pi\int_0^{\infty} re^{-r^2}dr \tag{0.10}$$
$$= -\pi\,e^{-r^2}\Big|_0^{\infty} \tag{0.11}$$
$$= \pi \tag{0.12}$$

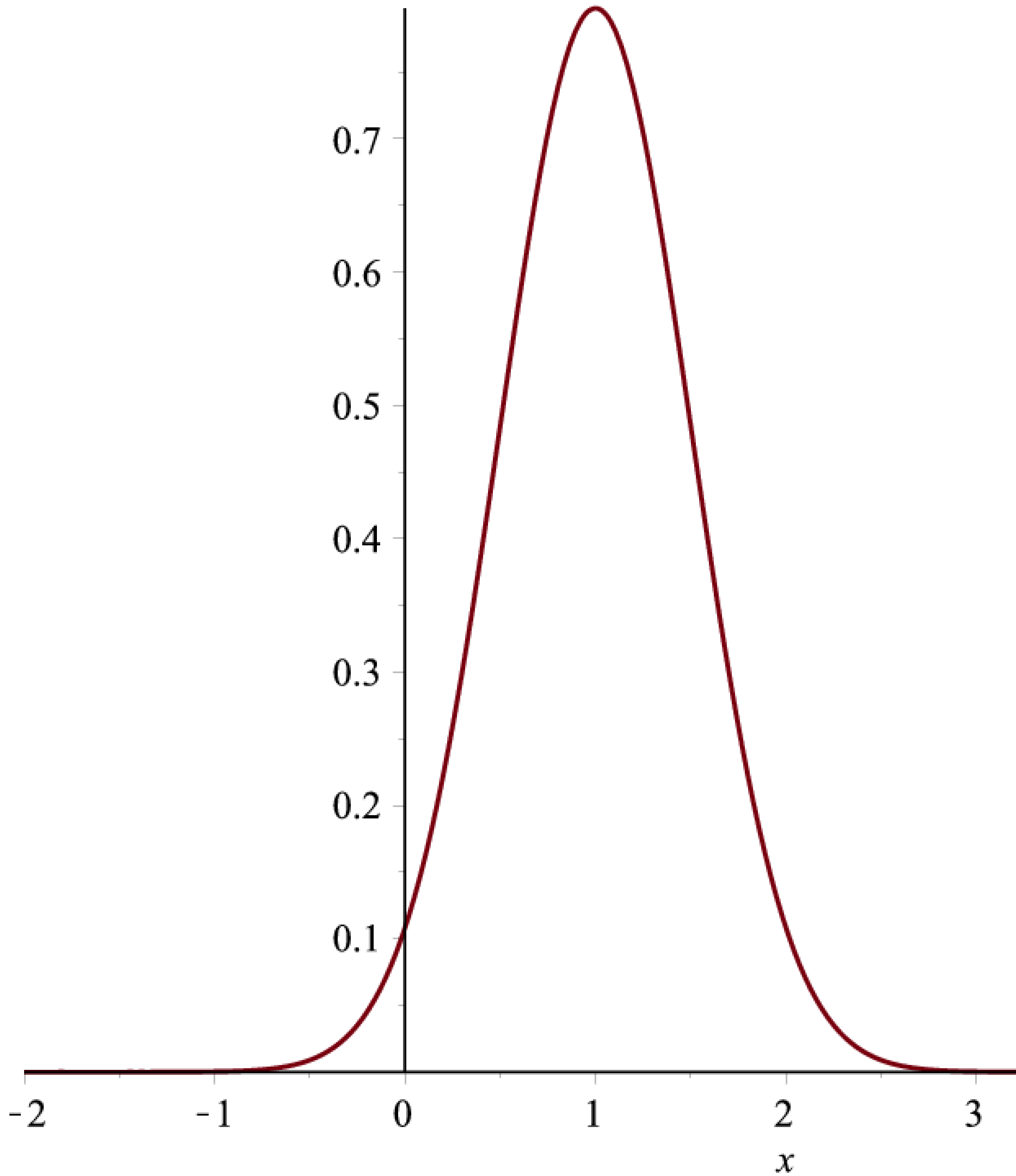
Therefore

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \tag{0.13}$$

Using Maple (or integration by parts) we can work out the average and variance.

$$\langle x\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = a \tag{0.14}$$
$$\left\langle x^2\right\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2e^{-\lambda(x-a)^2}dx = \frac{1}{2\lambda}+a^2 \tag{0.15}$$
$$\sigma^2 = \left\langle x^2\right\rangle - \langle x\rangle^2 = \frac{1}{2\lambda} \tag{0.16}$$

The distribution has the standard bell shape. Here's a plot for $\lambda = 2$ and $a = 1$:


0.7
0.6
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