

GAUSSIAN DISTRIBUTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.3.

Probably the most common continuous probability density is the gaussian distribution, specified by

$$\rho(x) = Ae^{-\lambda(x-a)^2} \quad (1)$$

First, we need to normalize the distribution by finding A . That is, we must have

$$A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = 1 \quad (2)$$

The gaussian integral is very common, and the result is that

$$\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2} dx = \sqrt{\frac{\pi}{\lambda}} \quad (3)$$

$$A = \sqrt{\frac{\lambda}{\pi}} \quad (4)$$

Although there is no closed form indefinite integral, the definite integral can be found by a cute trick.

$$\left[\int_{-\infty}^{\infty} e^{-x^2} dx \right]^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \quad (5)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy \quad (6)$$

We can now transform to polar coordinates using

$$r^2 = x^2 + y^2 \quad (7)$$

$$dx dy = r dr d\theta \quad (8)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \int_0^{\infty} \int_0^{2\pi} r e^{-r^2} d\theta dr \quad (9)$$

$$= 2\pi \int_0^{\infty} r e^{-r^2} dr \quad (10)$$

$$= -\pi e^{-r^2} \Big|_0^{\infty} \quad (11)$$

$$= \pi \quad (12)$$

Therefore

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (13)$$

Using Maple (or integration by parts) we can work out the average and variance.

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx = a \quad (14)$$

$$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx = \frac{1}{2\lambda} + a^2 \quad (15)$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda} \quad (16)$$

The distribution has the standard bell shape. Here's a plot for $\lambda = 2$ and $a = 1$:

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