HARMONIC OSCILLATOR: STATISTICS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.9.

Suppose a particle is in the quantum state

$$\Psi(x,t)=Ae^{-amx^2/\hbar}e^{-iat} \tag{1}$$

where A is the normalization constant and a is a constant with dimensions of 1/time. We can find A from normalization:

$$\begin{aligned}
\int_{-\infty}^{\infty}|\Psi|^2\,dx &= 1 && (2)\\
&= |A|^2\int_{-\infty}^{\infty}e^{-2amx^2/\hbar}dx && (3)\\
&= |A|^2\sqrt{\frac{\pi\hbar}{2ma}} && (4)\\
A &= \left(\frac{2ma}{\pi\hbar}\right)^{1/4} && (5)
\end{aligned}$$

The spatial component of the wave function is

$$\psi(x)=\left(\frac{2ma}{\pi\hbar}\right)^{1/4}e^{-amx^2/\hbar} \tag{6}$$

and it must satisfy the time-independent Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x) = E\psi(x) \tag{7}$$

The energy E can be found from the time equation:

$$i\hbar\frac{\partial\Xi}{\partial t}=E\Xi \tag{8}$$

where

$$\Xi(t) = e^{-iat} \tag{9}$$

Therefore

$$E = \hbar a \tag{10}$$

From 7 we have

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} a\left(\hbar - 2amx^2\right)e^{-amx^2/\hbar} \tag{11}$$

$$V(x) = \frac{E\psi(x) + \frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}}{\psi(x)} \tag{12}$$

$$= 2ma^2x^2 \tag{13}$$

This is the harmonic oscillator potential, and the wave function is actually the ground state of that potential.

We can work out a few average values:

$$\langle x\rangle = 0 \tag{14}$$

since $\psi(x)$ is even.

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty} x^2\psi^2 dx = \frac{\hbar}{4am} \tag{15}$$

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi\frac{\partial\psi}{\partial x}dx = 0 \tag{16}$$

$$\left\langle p^2\right\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi\frac{\partial^2\psi}{\partial x^2}dx = \hbar ma \tag{17}$$

The standard deviations are

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}\sqrt{\frac{\hbar}{ma}} \tag{18}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar ma} \tag{19}$$

and the uncertainty principle is

$$\sigma_x\sigma_p = \frac{\hbar}{2} \tag{20}$$

so in this case, the uncertainty is the minimum possible.