

## HARMONIC OSCILLATOR: STATISTICS

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Post date: 26 Jun 2015.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.9.

Suppose a particle is in the quantum state

$$\Psi(x, t) = Ae^{-amx^2/\hbar} e^{-iat} \quad (1)$$

where  $A$  is the normalization constant and  $a$  is a constant with dimensions of 1/time. We can find  $A$  from normalization:

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1 \quad (2)$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} dx \quad (3)$$

$$= |A|^2 \sqrt{\frac{\pi\hbar}{2ma}} \quad (4)$$

$$A = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} \quad (5)$$

The spatial component of the wave function is

$$\psi(x) = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} e^{-amx^2/\hbar} \quad (6)$$

and it must satisfy the time-independent Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad (7)$$

The energy  $E$  can be found from the time equation:

$$i\hbar \frac{\partial \Xi}{\partial t} = E\Xi \quad (8)$$

where

$$\Xi(t) = e^{-iat} \quad (9)$$

Therefore

$$E = \hbar a \quad (10)$$

From 7 we have

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = \left(\frac{2ma}{\pi\hbar}\right)^{1/4} a (\hbar - 2amx^2) e^{-amx^2/\hbar} \quad (11)$$

$$V(x) = \frac{E\psi(x) + \frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2}}{\psi(x)} \quad (12)$$

$$= 2ma^2x^2 \quad (13)$$

This is the harmonic oscillator potential, and the wave function is actually the ground state of that potential.

We can work out a few average values:

$$\langle x \rangle = 0 \quad (14)$$

since  $\psi(x)$  is even.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^2 dx = \frac{\hbar}{4am} \quad (15)$$

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi \frac{\partial\psi}{\partial x} dx = 0 \quad (16)$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi \frac{\partial^2\psi}{\partial x^2} dx = \hbar ma \quad (17)$$

The standard deviations are

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2} \sqrt{\frac{\hbar}{ma}} \quad (18)$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar ma} \quad (19)$$

and the uncertainty principle is

$$\sigma_x \sigma_p = \frac{\hbar}{2} \quad (20)$$

so in this case, the uncertainty is the minimum possible.