UNSTABLE PARTICLES: A CRUDE MODEL

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.15.

A rather unrealistic way of modelling an unstable particle is to introduce an imaginary component to the potential. We can see this by modifying the proof given in Griffiths's section 1.4 that, for a real potential, the normalization of the wave function is constant in time. We propose that

$$V(x) = V_0(x) - i\Gamma \tag{1}$$

where V_0 is the 'true' potential and Γ is a positive real constant.

The Schrödinger equation then says (where we're using a subscript t or x to denote a derivative with respect that variable):

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V_0\Psi - i\Gamma\Psi \tag{2}$$

$$-i\hbar\Psi_t^* = -\frac{\hbar^2}{2m}\Psi_{xx}^* + V_0\Psi^* + i\Gamma\Psi^* \tag{3}$$

where the second line is the complex conjugate of the first.

Retaining the interpretation of the wave function as a probability of finding the particle at a given place and time, we can calculate the time derivative of the total probability of finding the particle anywhere:

$$\frac{dP}{dt} \equiv \frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |\Psi|^2 dx \tag{4}$$

The derivative in the integrand is

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) \tag{5}$$

$$= \Psi_t^*\Psi + \Psi^*\Psi_t \tag{6}$$

From 2 and 3 we have

$$\Psi_t = i\frac{\hbar}{2m}\Psi_{xx} - \frac{i}{\hbar}V_0\Psi - \frac{\Gamma}{\hbar}\Psi \tag{7}$$

$$\Psi_t^* = -i\frac{\hbar}{2m}\Psi_{xx}^* + \frac{i}{\hbar}V_0\Psi^* - \frac{\Gamma}{\hbar}\Psi^* \tag{8}$$

$$\Psi_t^*\Psi + \Psi^*\Psi_t = i\frac{\hbar}{2m}\left(\Psi_{xx}\Psi^* - \Psi_{xx}^*\Psi\right) - 2\frac{\Gamma}{\hbar}\Psi^*\Psi \tag{9}$$

$$= i\frac{\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi_x\Psi^* - \Psi_x^*\Psi\right) - 2\frac{\Gamma}{\hbar}\Psi^*\Psi \tag{10}$$

Putting this into 4 we can integrate the first term and get zero because $\Psi \to 0$ as $x \to \pm\infty$ so we're left with

$$\frac{dP}{dt} = -2\frac{\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2\,dx = -2\frac{\Gamma}{\hbar}P \tag{11}$$

We can solve this differential equation to get

$$\frac{dP}{P} = -2\frac{\Gamma}{\hbar}dt \tag{12}$$

$$P = P_0e^{-2\Gamma t/\hbar} \tag{13}$$

where P_0 is the probability of finding the particle at $t = 0$. If we know the particle hasn't decayed at $t = 0$ then $P_0 = 1$.

The half-life of the particle is the time it takes for P to be reduced to $P_0/2$, so

$$\ln 0.5 = -2\frac{\Gamma}{\hbar}t_{1/2} \tag{14}$$

$$t_{1/2} = 0.347\frac{\hbar}{\Gamma} \tag{15}$$