INTEGRAL OF INNER PRODUCT OF TWO WAVE FUNCTIONS IS CONSTANT IN TIME

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.16.

The fact that the normalization of the wave function is constant over time is actually a special case of a more general theorem, which is

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = 0 \tag{1}$$

for any two normalizable solutions to the Schrödinger equation (with the same potential). The proof of this follows a similar derivation to that in section 1.4 of Griffiths's book.

The derivative in the integrand is (where we're using a subscript t or x to denote a derivative with respect that variable):

$$\frac{\partial}{\partial t}\left(\Psi_1^*\Psi_2\right) = \Psi_{1t}^*\Psi_2+\Psi_1^*\Psi_{2t} \tag{2}$$

From the Schrödinger equation

$$\Psi_{2t} = i\frac{\hbar}{2m}\Psi_{2xx}-\frac{i}{\hbar}V\Psi_2 \tag{3}$$

$$\Psi_{1t}^* = -i\frac{\hbar}{2m}\Psi_{1xx}^*+\frac{i}{\hbar}V\Psi_1^* \tag{4}$$

$$\Psi_{1t}^*\Psi_2+\Psi_1^*\Psi_{2t} = i\frac{\hbar}{2m}\left(-\Psi_{1xx}^*\Psi_2+\Psi_{2xx}\Psi_1^*\right)+\frac{i}{\hbar}V\left(\Psi_1^*\Psi_2-\Psi_1^*\Psi_2\right) \tag{5}$$

$$= i\frac{\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi_{2x}\Psi_1^*-\Psi_{1x}^*\Psi_2\right) \tag{6}$$

Inserting this into 1 and integrating gives zero because all wave functions go to zero at infinity. [Of course, the theorem doesn't hold if Ψ_1 and Ψ_2 are solutions for different potentials, because in that case the potential term wouldn't cancel out in 5.]