UNCERTAINTY PRINCIPLE: AN EXAMPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.17.

Here's another example of calculating the uncertainty principle. We have a wave function defined as

$$\Psi(x,0)=\begin{cases}A\left(a^2-x^2\right) & -a\le x\le a\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

The constant A is determined by normalization in the usual way:

$$\begin{aligned}
\int_{-a}^{a}|\Psi|^2\,dx &= 1 && (2)\\
&= A^2\left(\frac{x^5}{5}-\frac{2}{3}a^2x^3+a^4x\right)\bigg|_{-a}^{a} && (3)\\
&= A^2\frac{16a^5}{15} && (4)\\
A &= \frac{\sqrt{15}}{4a^{5/2}} && (5)
\end{aligned}$$

The expectation value of x is $\langle x\rangle=0$ from the symmetry of the wave function. The expectation value of p is

$$\begin{aligned}
\langle p\rangle &= -i\hbar\int_{-a}^{a}\Psi^*\frac{\partial}{\partial x}\Psi dx && (6)\\
&= -i\hbar\int_{-a}^{a}(-2Ax)A\left(a^2-x^2\right) && (7)\\
&= 0 && (8)
\end{aligned}$$

[We can't calculate $\langle p\rangle=\frac{d}{dt}(m\langle x\rangle)$ in this case, because we know the value of $\langle x\rangle$ only at one specific time ($t=0$), so we don't have enough information to calculate its derivative.]

The remaining statistics are (the integrals are all just integrals of polynomials, so nothing complicated):

$$\left\langle x^2\right\rangle = \int_{-a}^{a} x^2 \left|\Psi\right|^2 dx \tag{9}$$

$$= \frac{15}{16a^5}\left(\frac{x^7}{7} - \frac{2}{5}a^2x^5 + \frac{1}{3}a^4x^3\right)\Big|_{-a}^{a} \tag{10}$$

$$= \frac{a^2}{7} \tag{11}$$

$$\left\langle p^2\right\rangle = -\hbar^2 \int_{-a}^{a} \Psi^* \frac{\partial^2}{\partial x^2}\Psi dx \tag{12}$$

$$= \frac{15\hbar^2}{8a^5}\left(a^2 - x^2\right)\Big|_{-a}^{a} \tag{13}$$

$$= \frac{5\hbar^2}{2a^2} \tag{14}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \tag{15}$$

$$= \frac{a}{\sqrt{7}} \tag{16}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \tag{17}$$

$$= \sqrt{\frac{5}{2}}\frac{\hbar}{a} \tag{18}$$

$$\sigma_x\sigma_p = \sqrt{\frac{5}{14}}\hbar \tag{19}$$

$$\cong 0.598\hbar > \frac{\hbar}{2} \tag{20}$$

Thus the uncertainty principle is satisfied in this case.