

UNCERTAINTY PRINCIPLE: AN EXAMPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.17.

Here's another example of calculating the uncertainty principle. We have a wave function defined as

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2) & -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The constant A is determined by normalization in the usual way:

$$\int_{-a}^a |\Psi|^2 dx = 1 \quad (2)$$

$$= A^2 \left(\frac{x^5}{5} - \frac{2}{3}a^2x^3 + a^4x \right) \Big|_{-a}^a \quad (3)$$

$$= A^2 \frac{16a^5}{15} \quad (4)$$

$$A = \frac{\sqrt{15}}{4a^{5/2}} \quad (5)$$

The expectation value of x is $\langle x \rangle = 0$ from the symmetry of the wave function. The expectation value of p is

$$\langle p \rangle = -i\hbar \int_{-a}^a \Psi^* \frac{\partial}{\partial x} \Psi dx \quad (6)$$

$$= -i\hbar \int_{-a}^a (-2Ax) A (a^2 - x^2) dx \quad (7)$$

$$= 0 \quad (8)$$

[We can't calculate $\langle p \rangle = \frac{d}{dt} (m\langle x \rangle)$ in this case, because we know the value of $\langle x \rangle$ only at one specific time ($t = 0$), so we don't have enough information to calculate its derivative.]

The remaining statistics are (the integrals are all just integrals of polynomials, so nothing complicated):

$$\langle x^2 \rangle = \int_{-a}^a x^2 |\Psi|^2 dx \quad (9)$$

$$= \frac{15}{16a^5} \left(\frac{x^7}{7} - \frac{2}{5}a^2x^5 + \frac{1}{3}a^4x^3 \right) \Big|_{-a}^a \quad (10)$$

$$= \frac{a^2}{7} \quad (11)$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-a}^a \Psi^* \frac{\partial^2}{\partial x^2} \Psi dx \quad (12)$$

$$= \frac{15\hbar^2}{8a^5} (a^2 - x^2) \Big|_{-a}^a \quad (13)$$

$$= \frac{5\hbar^2}{2a^2} \quad (14)$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad (15)$$

$$= \frac{a}{\sqrt{7}} \quad (16)$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \quad (17)$$

$$= \sqrt{\frac{5}{2}} \frac{\hbar}{a} \quad (18)$$

$$\sigma_x \sigma_p = \sqrt{\frac{5}{14}} \hbar \quad (19)$$

$$\cong 0.598\hbar > \frac{\hbar}{2} \quad (20)$$

Thus the uncertainty principle is satisfied in this case.