UNCERTAINTY PRINCIPLE: AN EXAMPLE

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 1.17.

Here's another example of calculating the uncertainty principle. We have a wave function defined as

$$\Psi(x,0)=\begin{cases}A\left(a^{2}-x^{2}\right) & -a\le x\le a\\0 & \text{otherwise}\end{cases}\tag{1}$$

The constant A is determined by normalization in the usual way:

$$\begin{aligned}\int_{-a}^{a}|\Psi|^{2}dx &= 1 &(2)\\ &= A^{2}\left(\frac{x^{5}}{5}-\frac{2}{3}a^{2}x^{3}+a^{4}x\right)\Big|_{-a}^{a} &(3)\\ &= A^{2}\frac{16a^{5}}{15} &(4)\\ A &= \frac{\sqrt{15}}{4a^{5/2}} &(5)\end{aligned}$$

The expectation value of x is $\langle x\rangle=0$ from the symmetry of the wave function. The expectation value of p is

$$\begin{aligned}\langle p\rangle &= -i\hbar\int_{-a}^{a}\Psi^{*}\frac{\partial}{\partial x}\Psi dx &(6)\\ &= -i\hbar\int_{-a}^{a}(-2Ax)A\left(a^{2}-x^{2}\right) &(7)\\ &= 0 &(8)\end{aligned}$$

[We can't calculate $\langle p\rangle=\frac{d}{dt}(m\langle x\rangle)$ in this case, because we know the value of $\langle x\rangle$ only at one specific time ($t=0$), so we don't have enough information to calculate its derivative.]

The remaining statistics are (the integrals are all just integrals of polynomials, so nothing complicated):

$$\begin{aligned}
\left\langle x^{2}\right\rangle &= \int_{-a}^{a} x^{2}\left|\Psi\right|^{2} dx && (9)\\
&= \frac{15}{16a^{5}}\left(\frac{x^{7}}{7}-\frac{2}{5}a^{2}x^{5}+\frac{1}{3}a^{4}x^{3}\right)\Big|_{-a}^{a} && (10)\\
&= \frac{a^{2}}{7} && (11)\\
\left\langle p^{2}\right\rangle &= -\hbar^{2}\int_{-a}^{a}\Psi^{*}\frac{\partial^{2}}{\partial x^{2}}\Psi dx && (12)\\
&= \frac{15\hbar^{2}}{8a^{5}}\left(a^{2}-x^{2}\right)\Big|_{-a}^{a} && (13)\\
&= \frac{5\hbar^{2}}{2a^{2}} && (14)\\
\sigma_{x} &= \sqrt{\langle x^{2}\rangle-\langle x\rangle^{2}} && (15)\\
&= \frac{a}{\sqrt{7}} && (16)\\
\sigma_{p} &= \sqrt{\langle p^{2}\rangle-\langle p\rangle^{2}} && (17)\\
&= \sqrt{\frac{5}{2}}\frac{\hbar}{a} && (18)\\
\sigma_{x}\sigma_{p} &= \sqrt{\frac{5}{14}}\hbar && (19)\\
&\cong 0.598\hbar>\frac{\hbar}{2} && (20)
\end{aligned}$$

Thus the uncertainty principle is satisfied in this case.