HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 17 Jul 2012.

Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.12.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.2.

In the study of the harmonic oscillator, we can express x and p in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \tag{1}$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \tag{2}$$

We now have

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int\psi_n^*(a_+ + a_-)\psi_n dx \tag{3}$$

$$= 0 \tag{4}$$

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \tag{5}$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1} \tag{6}$$

$$a_+a_-\psi_n = n\psi_n \tag{7}$$

$$a_-a_+\psi_n = (n+1)\psi_n \tag{8}$$

and since the wave functions are orthogonal, we get

$$\int\psi_n^*\psi_{n+1}dx = \int\psi_n^*\psi_{n-1}dx = 0 \tag{9}$$

Similarly:

$$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\int\psi_n^*(a_+-a_-)\psi_n dx \tag{10}$$

$$= 0 \tag{11}$$

for the same reason.

For the mean squares:

$$\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_++a_-)(a_++a_-)\psi_n dx \tag{12}$$

$$= \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_+a_-+a_-a_+)\psi_n dx \tag{13}$$

$$= \left(\frac{\hbar}{2m\omega}\right)(2n+1) \tag{14}$$

$$= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \tag{15}$$

In going from the first to the second line, we've thrown out terms where we integrate two orthogonal functions. For example,

$$\int\psi_n^*a_+a_+\psi_n dx = \int\psi_n^*\sqrt{(n+1)(n+2)}\psi_{n+2}dx \tag{16}$$

$$= 0 \tag{17}$$

We have used the relations above and the fact that ψ_n is normalized to get the third line.

Similarly:

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\int\psi_n^*(-a_+a_--a_-a_+)\psi_n dx \tag{18}$$

$$= \hbar m\omega\left(n+\frac{1}{2}\right) \tag{19}$$

The uncertainty principle then becomes

$$\sigma_p\sigma_x = \sqrt{\langle x^2\rangle\langle p^2\rangle} \tag{20}$$

$$= \hbar\left(n+\frac{1}{2}\right) \tag{21}$$

and the kinetic energy is

$$\langle T\rangle=\frac{\langle p^{2}\rangle}{2m}=\frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) \tag{22}$$

which is half the total energy, as it should be.

PINGBACKS

Pingback: Harmonic oscillator - mixed initial state
Pingback: Harmonic oscillator: matrix elements
Pingback: Harmonic oscillator: coherent states
Pingback: Harmonic oscillator: relativistic correction
Pingback: Forbidden transitions in the harmonic oscillator and hydrogen
Pingback: Adiabatic approximation: higher order corrections
Pingback: Virial theorem in classical mechanics; application to harmonic oscillator