HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.12.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.2.

In the study of the harmonic oscillator, we can express x and p in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_{+}+a_{-}) \tag{1}$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_{+}-a_{-}) \tag{2}$$

We now have

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int\psi_n^*(a_{+}+a_{-})\psi_n dx \tag{3}$$

$$= 0 \tag{4}$$

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_{+}\psi_n = \sqrt{n+1}\psi_{n+1} \tag{5}$$

$$a_{-}\psi_n = \sqrt{n}\psi_{n-1} \tag{6}$$

$$a_{+}a_{-}\psi_n = n\psi_n \tag{7}$$

$$a_{-}a_{+}\psi_n = (n+1)\psi_n \tag{8}$$

and since the wave functions are orthogonal, we get

$$\int\psi_n^*\psi_{n+1}dx = \int\psi_n^*\psi_{n-1}dx = 0 \tag{9}$$

Similarly:

$$\begin{aligned}\langle p\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\int\psi_n^*(a_+-a_-)\psi_n dx &(10)\\ &= 0 &(11)\end{aligned}$$

for the same reason.

For the mean squares:

$$\begin{aligned}\langle x^2\rangle &= \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_++a_-)(a_++a_-)\psi_n dx &(12)\\ &= \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_+a_-+a_-a_+)\psi_n dx &(13)\\ &= \left(\frac{\hbar}{2m\omega}\right)(2n+1) &(14)\\ &= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) &(15)\end{aligned}$$

In going from the first to the second line, we've thrown out terms where we integrate two orthogonal functions. For example,

$$\begin{aligned}\int\psi_n^*a_+a_+\psi_n dx &= \int\psi_n^*\sqrt{(n+1)(n+2)}\psi_{n+2}dx &(16)\\ &= 0 &(17)\end{aligned}$$

We have used the relations above and the fact that ψ_n is normalized to get the third line.

Similarly:

$$\begin{aligned}\langle p^2\rangle &= -\frac{\hbar m\omega}{2}\int\psi_n^*(-a_+a_--a_-a_+)\psi_n dx &(18)\\ &= \hbar m\omega\left(n+\frac{1}{2}\right) &(19)\end{aligned}$$

The uncertainty principle then becomes

$$\begin{aligned}\sigma_p\sigma_x &= \sqrt{\langle x^2\rangle\langle p^2\rangle} &(20)\\ &= \hbar\left(n+\frac{1}{2}\right) &(21)\end{aligned}$$

and the kinetic energy is

$$\langle T\rangle=\frac{\langle p^{2}\rangle}{2m}=\frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) \tag{22}$$

which is half the total energy, as it should be.

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