HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.12.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.2.

In the study of the harmonic oscillator, we can express x and p in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \tag{0.1}$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \tag{0.2}$$

We now have

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int\psi_n^*(a_+ + a_-)\psi_n dx \tag{0.3}$$

$$= 0 \tag{0.4}$$

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \tag{0.5}$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1} \tag{0.6}$$

$$a_+a_-\psi_n = n\psi_n \tag{0.7}$$

$$a_-a_+\psi_n = (n+1)\,\psi_n \tag{0.8}$$

and since the wave functions are orthogonal, we get

$$\int\psi_n^*\psi_{n+1}dx = \int\psi_n^*\psi_{n-1}dx = 0 \tag{0.9}$$

Similarly:

$$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\int\psi_n^*(a_+ - a_-)\psi_n dx \tag{0.10}$$
$$= 0 \tag{0.11}$$

for the same reason.

For the mean squares:

$$\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_+ + a_-)(a_+ + a_-)\psi_n dx \tag{0.12}$$
$$= \left(\frac{\hbar}{2m\omega}\right)\int\psi_n^*(a_+a_- + a_-a_+)\psi_n dx \tag{0.13}$$
$$= \left(\frac{\hbar}{2m\omega}\right)(2n+1) \tag{0.14}$$
$$= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \tag{0.15}$$

In going from the first to the second line, we've thrown out terms where we integrate two orthogonal functions. For example,

$$\int\psi_n^* a_+a_+\psi_n dx = \int\psi_n^*\sqrt{(n+1)(n+2)}\psi_{n+2}dx \tag{0.16}$$
$$= 0 \tag{0.17}$$

We have used the relations above and the fact that ψ_n is normalized to get the third line.

Similarly:

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\int\psi_n^*(-a_+a_- - a_-a_+)\psi_n dx \tag{0.18}$$
$$= \hbar m\omega\left(n+\frac{1}{2}\right) \tag{0.19}$$

The uncertainty principle then becomes

$$\sigma_p\sigma_x = \sqrt{\langle x^2\rangle\langle p^2\rangle} \tag{0.20}$$
$$= \hbar\left(n+\frac{1}{2}\right) \tag{0.21}$$

and the kinetic energy is

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) \tag{0.22}$$

which is half the total energy, as it should be.

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