

## GRAM-SCHMIDT ORTHOGONALIZATION - A COUPLE OF EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.3.1 - 1.3.2.

Here are a couple of examples of the Gram-Schmidt orthogonalization procedure. The recipe for generating an orthonormal basis  $e_i$  from a general set of linearly independent vectors  $v_i$  is as follows.

The first vector  $e_1$  in the orthonormal basis is defined by

$$e_1 = \frac{v_1}{|v_1|} \quad (1)$$

where  $v_1$  is the first vector (well, any vector, really) in the non-orthonormal basis.

Given vector  $e_{j-1}$  in the orthonormal basis, we can form  $e_j$  from the formula

$$e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \quad (2)$$

**Example 1.** Given  $v_1 = (3, 4)$  and  $v_2 = (2, -6)$  we can form an orthonormal basis in two ways. Starting with  $v_1$  we have

$$e_1 = \frac{v_1}{|v_1|} = \left( \frac{3}{5}, \frac{4}{5} \right) \quad (3)$$

$$e_2 = \frac{v_2 - \langle e_1, v_2 \rangle e_1}{|v_2 - \langle e_1, v_2 \rangle e_1|} \quad (4)$$

To evaluate  $e_2$ , we have

$$\langle e_1, v_2 \rangle = \frac{6}{5} - \frac{24}{5} = -\frac{18}{5} \quad (5)$$

$$v_2 - \langle e_1, v_2 \rangle e_1 = (2, -6) + \frac{18}{5} \left( \frac{3}{5}, \frac{4}{5} \right) \quad (6)$$

$$= \frac{1}{25} (104, -78) \quad (7)$$

$$|v_2 - \langle e_1, v_2 \rangle e_1| = \frac{130}{25} \quad (8)$$

$$e_2 = \frac{1}{130} (104, -78) \quad (9)$$

As a check,

$$\langle e_1, e_2 \rangle = \frac{1}{650} (312 - 312) = 0 \quad (10)$$

$$\langle e_1, e_1 \rangle = \frac{1}{25} (9 + 16) = 1 \quad (11)$$

$$\langle e_2, e_2 \rangle = \frac{1}{16900} (10816 + 6084) = 1 \quad (12)$$

We could also start with  $v_2$ , giving

$$e_1 = \frac{v_2}{|v_2|} = \frac{1}{2\sqrt{10}} (2, -6) \quad (13)$$

$$e_2 = \frac{v_1 - \langle e_1, v_1 \rangle e_1}{|v_1 - \langle e_1, v_1 \rangle e_1|} \quad (14)$$

$$\langle e_1, v_1 \rangle = \frac{1}{2\sqrt{10}} (6 - 24) = -\frac{9}{\sqrt{10}} \quad (15)$$

$$v_1 - \langle e_1, v_1 \rangle e_1 = (3, 4) + \frac{9}{20} (2, -6) \quad (16)$$

$$= \frac{1}{20} (78, 26) \quad (17)$$

$$|v_1 - \langle e_1, v_1 \rangle e_1| = \frac{\sqrt{6760}}{20} \quad (18)$$

$$e_2 = \frac{1}{\sqrt{6760}} (78, 26) \quad (19)$$

Checking, we get

$$\langle e_1, e_2 \rangle = \frac{1}{2\sqrt{67600}} (156 - 156) = 0 \quad (20)$$

$$\langle e_1, e_1 \rangle = \frac{1}{40} (4 + 36) = 1 \quad (21)$$

$$\langle e_2, e_2 \rangle = \frac{1}{6760} (6084 + 676) = 1 \quad (22)$$

**Example 2.** We're now given 3 vectors in 3-d space:

$$v_1 = (3, 0, 0) \quad (23)$$

$$v_2 = (0, 1, 2) \quad (24)$$

$$v_3 = (0, 2, 5) \quad (25)$$

The problem is to generate linear combinations of these 3 vectors to give the orthonormal basis

$$e_1 = (1, 0, 0) \quad (26)$$

$$e_2 = \frac{1}{\sqrt{5}} (0, 1, 2) \quad (27)$$

$$e_3 = \frac{1}{\sqrt{5}} (0, -2, 1) \quad (28)$$

We could use the Gram-Schmidt procedure, but it's probably easier to just solve the equations. We have

$$e_1 = \frac{v_1}{3} \quad (29)$$

$$e_2 = \frac{v_2}{\sqrt{5}} \quad (30)$$

$$e_3 = Av_1 + Bv_2 + Cv_3 \quad (31)$$

Writing out the last equation using components, we have

$$0 = A \quad (32)$$

$$-\frac{2}{\sqrt{5}} = B + 2C \quad (33)$$

$$\frac{1}{\sqrt{5}} = 2B + 5C \quad (34)$$

The solution is

$$C = \frac{5}{\sqrt{5}} = \sqrt{5} \quad (35)$$

$$B = -\frac{12}{\sqrt{5}} \quad (36)$$

Thus we have

$$e_1 = \frac{v_1}{3} \quad (37)$$

$$e_2 = \frac{v_2}{\sqrt{5}} \quad (38)$$

$$e_3 = -\frac{12}{\sqrt{5}}v_2 + \sqrt{5}v_3 \quad (39)$$