TRIANGLE INEQUALITY AS AN EQUALITY

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.3.3 - 1.3.4.

We've already proved the triangle inequality for vectors, but it's worth adding a note on when the inequality becomes an equality. The triangle inequality states that for all $u,v\in V$

$$|u+v|\le|u|+|v| \tag{1}$$

To make this an equality, we need to look back at the proof. The last step in the proof invokes the Schwarz inequality to state that

$$|u+v|^2\le|u|^2+|v|^2+2|u||v| \tag{2}$$

Looking at the proof for the Schwarz inequality, we see that it becomes an equality if the component w of u that is orthogonal to v is zero, that is, if $u=\alpha v$ for some (possibly complex) scalar α. If that is the case, then

$$|u+v| = |\alpha v+v|=|1+\alpha||v| \tag{3}$$

$$|u|+|v| = |\alpha v|+|v|=(1+|\alpha|)|v| \tag{4}$$

Thus the triangle inequality becomes an equality if

$$|1+\alpha|=1+|\alpha| \tag{5}$$

which occurs if α is real and $\alpha\ge0$. In terms of vectors as arrows in 3-d space, this condition is equivalent to the two vectors being parallel and pointing in the same direction (rather than in opposite directions).

To see that equality doesn't happen if α is complex, suppose $\alpha=1+i$. Then

$$|1+\alpha| = |2+i|=\sqrt{5} \tag{6}$$

$$1+|\alpha| = 1+\sqrt{2} \tag{7}$$