

## TRIANGLE INEQUALITY AS AN EQUALITY

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.3.3 - 1.3.4.

We've already proved the triangle inequality for vectors, but it's worth adding a note on when the inequality becomes an equality. The triangle inequality states that for all  $u, v \in V$

$$(1) \quad |u + v| \leq |u| + |v|$$

To make this an equality, we need to look back at the proof. The last step in the proof invokes the Schwarz inequality to state that

$$(2) \quad |u + v|^2 \leq |u|^2 + |v|^2 + 2|u||v|$$

Looking at the proof for the Schwarz inequality, we see that it becomes an equality if the component  $w$  of  $u$  that is orthogonal to  $v$  is zero, that is, if  $u = \alpha v$  for some (possibly complex) scalar  $\alpha$ . If that is the case, then

$$(3) \quad |u + v| = |\alpha v + v| = |1 + \alpha| |v|$$

$$(4) \quad |u| + |v| = |\alpha v| + |v| = (1 + |\alpha|) |v|$$

Thus the triangle inequality becomes an equality if

$$(5) \quad |1 + \alpha| = 1 + |\alpha|$$

which occurs if  $\alpha$  is real and  $\alpha \geq 0$ . In terms of vectors as arrows in 3-d space, this condition is equivalent to the two vectors being parallel and pointing in the same direction (rather than in opposite directions).

To see that equality doesn't happen if  $\alpha$  is complex, suppose  $\alpha = 1 + i$ . Then

$$(6) \quad |1 + \alpha| = |2 + i| = \sqrt{5}$$

$$(7) \quad 1 + |\alpha| = 1 + \sqrt{2}$$