VECTOR SPACES - NUMBER OF DIMENSIONS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.4.1 - 1.4.2.

Here are a couple of theorems that arise from the subspace theorem we proved earlier, which is:

If U is a subspace of V, then $V = U \oplus U^{\perp}$. (Recall the direct sum.) Here, the orthogonal complement $U^{\perp}$ of U is the set of all vectors that are orthogonal to all vectors $u \in U$.

First, we can show that:

Theorem 1. *The dimensionality of a vector space is $n_{\perp}$, the maximum number of mutually orthogonal vectors in the space.*

Proof. The set of mutually orthogonal vectors is linearly independent, and since it is the largest such set, any vector $v \in V$ can be written as a linear combination of them. Thus the dimension of the space cannot be greater than $n_{\perp}$. Since the set is linearly dependent, no member of the set can be written as a linear combination of the remaining members of the set, so the dimension can't be less than $n_{\perp}$. Thus the dimension must be equal to $n_{\perp}$. □

Now we look at a couple of other theorems.

Theorem 2. *In a vector space V^n of dimension n, the set $V_{\perp}$ of all vectors orthogonal to any specific vector $v \neq |0\rangle$ forms a subspace V^{n-1} of dimension $n-1$.*

Proof. From the subspace theorem above, if we take U to be the subspace spanned by v, then $U^{\perp}$ is the orthogonal subspace. Since the dimension of U is 1 and $V^n = U \oplus U^{\perp}$, the dimension of $U^{\perp} = V^{n-1}$ is $n-1$. □

Theorem 3. *Given two subspaces $V_1^{n_1}$ and $V_2^{n_2}$ such that every vector $v_1 \in V_1$ is orthogonal to every vector $v_2 \in V_2$, the dimension of $V_1 \oplus V_2$ is $n_1 + n_2$.*

Proof. An orthonormal basis of V_1 consists of n_1 mutually orthogonal vectors in V_1, and similarly, an orthonormal basis of V_2 consists of n_2 mutually orthogonal vectors in V_2. These bases consist of the maximum number of mutually orthogonal vectors in their respective spaces. In the direct sum $V_1 \oplus V_2$, we therefore have a set of $n_1 + n_2$ mutually orthogonal vectors, which is the maximum number of such vectors in $V_1 \oplus V_2$. This follows because a vector $w \in V_1 \oplus V_2$ must be a linear combination of a vector $v_1 \in V_1$ and a vector $v_2 \in V_2$, where v_i is, in turn, a linear combination of the basis of space V_i. Thus $w = v_1 + v_2$ must be a linear combination of vectors from the two bases combined. Hence the dimension of $V_1 \oplus V_2$ is $n_1 + n_2$. $\square$