UNITARY MATRICES - SOME EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.6.3 - 1.6.6.

Here are a few more results about unitary operators.

Shankar defines a unitary operator U as one where

$$UU^{\dagger}=I \tag{1}$$

From this we can derive the other condition by which they can be defined, namely that a unitary operator preserves the norm of a vector:

$$|Uv|=|v| \tag{2}$$

This follows, for if we define the effect of U by

$$\left|v_1'\right\rangle = U\left|v_1\right\rangle \tag{3}$$

then

$$\begin{aligned}
\left\langle v_1' \middle| v_1' \right\rangle &= \left\langle Uv_1 \middle| Uv_1 \right\rangle && (4)\\
&= \left\langle v_1 \middle| U^{\dagger}Uv_1 \right\rangle && (5)\\
&= \left\langle v_1 \middle| v_1 \right\rangle && (6)
\end{aligned}$$

Thus $|v_1'|^2=|v_1|^2$.

Theorem 1. *The product of two unitary operators U_1 and U_2 is unitary.*

Proof. Using Shankar's definition 1, we have

$$\begin{aligned}
\left(U_1U_2\right)^{\dagger}U_1U_2 &= U_2^{\dagger}U_1^{\dagger}U_1U_2 && (7)\\
&= U_2^{\dagger}IU_2 && (8)\\
&= U_2^{\dagger}U_2 && (9)\\
&= I && (10)
\end{aligned}$$

□

Theorem 2. *The determinant of a unitary matrix U is a complex number with unit modulus.*

Proof. The determinant of a hermitian conjugate is the complex conjugate of the determinant of the original matrix, since $\det U = \det U^T$ (where the superscript T denotes the transpose) for any matrix, and the hermitian conjugate is the complex conjugate transpose. Therefore

$$\det\left(UU^\dagger\right) = [\det U]\,[\det U]^* = \det I = 1 \tag{11}$$

Therefore $|\det U|^2 = 1$ as required. $\square$

Example 3. The rotation matrix $R\left(\frac{\pi}{2}\mathbf{i}\right)$ is unitary. We have

$$R\left(\frac{\pi}{2}\mathbf{i}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{12}$$

By direct calculation

$$RR^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \tag{14}$$

Example 4. Consider the matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \tag{15}$$

By calculating

$$UU^\dagger = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \tag{16}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I \tag{17}$$

Thus U is unitary, but because $U \neq U^\dagger$ it is not hermitian. Its determinant is

$$\det U = \left(\frac{1}{\sqrt{2}}\right)^2 \left(1 - i^2\right) = 1 \tag{18}$$

This is of the required form $e^{i\theta}$ with $\theta = 0$.

Example 5. Consider the matrix

$$U=\frac{1}{2}\left[\begin{array}{cc}1+i & 1-i\\ 1-i & 1+i\end{array}\right] \tag{19}$$

$$UU^{\dagger}=\frac{1}{4}\left[\begin{array}{cc}1+i & 1-i\\ 1-i & 1+i\end{array}\right]\left[\begin{array}{cc}1-i & 1+i\\ 1+i & 1-i\end{array}\right] \tag{20}$$

$$=\frac{1}{4}\left[\begin{array}{cc}4 & 0\\ 0 & 4\end{array}\right]=I \tag{21}$$

Thus U is unitary, but because $U\ne U^{\dagger}$ it is not hermitian. Its determinant is

$$\det U = \left(\frac{1}{2}\right)^{2}\left[(1+i)^{2}-(1-i)^{2}\right] \tag{22}$$

$$= i \tag{23}$$

This is of the required form $e^{i\theta}$ with $\theta=\frac{\pi}{2}$.

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