

UNITARY OPERATORS: ACTIVE AND PASSIVE TRANSFORMATIONS OF AN OPERATOR

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.7.1 - 1.7.2.

A unitary operator transforms one orthonormal basis to another. Therefore if we have an operator Ω with matrix elements $\Omega_{ij}(\{f\}) = \langle f_i | \Omega | f_j \rangle$ in one orthonormal basis (f_1, \dots, f_n) , we can transform the basis to another orthonormal basis (e_1, \dots, e_n) by a unitary transformation U so that

$$|e_i\rangle = U |f_i\rangle \quad (1)$$

This results in a transformation of the operator Ω 's matrix elements:

$$\Omega_{ij}(\{e\}) = \langle e_i | \Omega | e_j \rangle \quad (2)$$

$$= \langle U f_i | \Omega | U f_j \rangle \quad (3)$$

$$= \langle f_i | U^\dagger \Omega U | f_j \rangle \quad (4)$$

Thus we can view the transformation as either a transformation of the basis vectors $e_i = U f_i$, known as an *active transformation*, or as a transformation of the operator according to $\Omega \rightarrow U^\dagger \Omega U$, known as a *passive transformation*. The matrix elements of $U^\dagger \Omega U$ in the original basis $\{f\}$ are equal to the matrix elements of the original operator Ω in the new basis $\{e\}$.

We've already seen a few results about the trace and determinant of products of matrices. We'll list these here for reference:

- $\text{Tr}(\Omega\Lambda) = \text{Tr}(\Lambda\Omega)$. That is, even if the operators don't commute, the trace of a product of operators doesn't depend on the order of the operators in the product.
- The trace of a product of 3 or more operators is invariant under cyclic permutation. $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Lambda\theta\Omega) = \text{Tr}(\theta\Omega\Lambda)$. This follows directly from the previous result. For example, we can define $A \equiv \Lambda\theta$ so that $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Omega A) = \text{Tr}(A\Omega) = \text{Tr}(\Lambda\theta\Omega)$.
- The determinant of a unitary matrix is a complex number with modulus 1.

We can use these to prove a couple of further results about unitary transformations.

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The trace of an operator is invariant under a unitary transformation:

$$\mathrm{Tr}\left(U^\dagger \Omega U\right) = \mathrm{Tr}\left(UU^\dagger \Omega\right) = \mathrm{Tr}\Omega \quad (5)$$

since $UU^\dagger = I$.

Finally, the determinant of an operator is also invariant under a unitary transformation. Since the determinant of a product is the product of the determinants,

$$\det\left(U^\dagger \Omega U\right) = \det U^\dagger \det \Omega \det U \quad (6)$$

$$= e^{-i\alpha} \det \Omega e^{i\alpha} \quad (7)$$

$$= \det \Omega \quad (8)$$

In the second line we used the fact that $\det U$ is a complex number with unit modulus, and the fact that $\det U^\dagger = (\det U)^*$.

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