EIGENVALUES AND EIGENVECTORS - EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.8.1 - 1.8.4.

Here are a few examples of calculating eigenvalues and eigenvectors.

Example 1. Find the eigenvalues and normalized eigenvectors of

$$\Omega=\left[\begin{array}{ccc}1 & 3 & 1\\0 & 2 & 0\\0 & 1 & 4\end{array}\right] \tag{1}$$

The eigenvalues are solutions of $\det(\Omega-\lambda I)=0$ which gives, calculating the determinant down the first column:

$$\begin{aligned}(1-\lambda)(2-\lambda)(4-\lambda) & = 0 \quad (2)\\ \lambda & = 1,2,4 \quad (3)\end{aligned}$$

The eigenvectors v_i satisfy $(\Omega-\lambda_iI)v_i=0v_i$ for each eigenvalue λ_i. We get, for $\lambda_1=1$:

$$\left[\begin{array}{ccc}0 & 3 & 1\\0 & 1 & 0\\0 & 1 & 3\end{array}\right]\left[\begin{array}{c}a\\b\\c\end{array}\right]=\left[\begin{array}{c}0\\0\\0\end{array}\right] \tag{4}$$

Solving, we find

$$\begin{aligned}b & = c=0 \quad (5)\\ a & = \text{anything} \quad (6)\end{aligned}$$

Thus a normalized eigenvector is

$$v_1=\left[\begin{array}{c}1\\0\\0\end{array}\right] \tag{7}$$

For $\lambda_2=2$, we have

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Solving:

$$\begin{aligned} b &= -2c && (9) \\ a &= 3b+c && (10) \\ &= -5c && (11) \end{aligned}$$

Choosing $c = 1$ and normalizing, we have

$$v_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \tag{12}$$

Finally, for $\lambda_3 = 4$ we have

$$\begin{bmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Solving:

$$\begin{aligned} b &= 0 && (14) \\ 3a &= 3b+c && (15) \\ &= c && (16) \end{aligned}$$

Choosing $a = 1$ and normalizing:

$$v_3 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \tag{17}$$

The matrix Ω is not Hermitian since $\Omega^\dagger \neq \Omega$, and we can see by inspection that the eigenvectors are not orthogonal.

Example 2. Now we have

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{18}$$

It is hermitian since $\Omega^\dagger = \Omega^T = \Omega$. The eigenvalues are found from

$$(-\lambda)^3 + \lambda = 0 \tag{19}$$
$$\lambda = 0, -1, 1 \tag{20}$$

Solving for the eigenvectors in the same way as in the last example, we get, for $\lambda_i = 0, -1, 1$ in that order:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{21}$$
$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$
$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

The eigenvectors are orthogonal, as required for a hermitian matrix. We can diagonalize Ω by means of a unitary transformation U, where the columns of U are the eigenvectors of Ω. We have

$$U = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{24}$$
$$U^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{25}$$

We can verify by direct matrix multiplication that

$$U^\dagger \Omega U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Note that the order of eigenvalues in the diagonal is determined by the order in which we place the columns in U.

Example 3. We now have the hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \tag{27}$$

The eigenvectors follow from

$$(1-\lambda)\left[\left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4}\right] = 0 \tag{28}$$

$$\lambda = 1,1,2 \tag{29}$$

Thus the eigenvalue $\lambda = 1$ is degenerate. We can find the eigenvector corresponding to $\lambda_3 = 2$ in the usual way and get

$$v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{30}$$

The other two eigenvectors span a 2-d subspace that must be orthogonal to v_3 (since Ω is hermitian; in the more general case, the orthogonality is not guaranteed). We can therefore find two vectors v_1, v_2 in the subspace by requiring $\langle v_1, v_3\rangle = \langle v_2, v_3\rangle = 0$. That is, if

$$v_{1,2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{31}$$

we must have

$$a = \text{anything} \tag{32}$$

$$b = c \tag{33}$$

These two equations can be satisfied by a variety of v_1 and v_2, but if we want $\langle v_1, v_2\rangle = 0$ as well, we can choose $a = 1$ and $b = c = 1$, then normalize, to get

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{35}$$

The 2-d subspace spanned by v_1 and v_2 is therefore

$$v = av_1 + bv_2 = \begin{bmatrix} a \\ b \\ b \end{bmatrix} \tag{36}$$

Thus any normalized eigenvector of $\lambda = 1$ has the form

$$e = \frac{1}{\sqrt{a^2+2b^2}} \begin{bmatrix} a \\ b \\ b \end{bmatrix} \tag{37}$$

Example 4. Now let's look at a non-hermitian matrix:

$$\Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \tag{38}$$

The eigenvalues are found from

$$\begin{aligned} (4-\lambda)(2-\lambda)+1 &= 0 && (39)\\ (\lambda-3)^2 &= 0 && (40)\\ \lambda &= 3,3 && (41) \end{aligned}$$

Thus there is one degenerate eigenvalue. To find the eigenvector(s), we solve $(\Omega - \lambda I)v = 0$ as usual:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \tag{42}$$

This gives only one condition, namely $a = -b$. Thus there is only one normalized eigenvector:

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{43}$$