HERMITIAN MATRICES - EXAMPLE WITH 4 MATRICES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.8.

Suppose we have four hermitian matrices M^i for $i = 1,2,3,4$ that obey the relation

$$M^i M^j + M^j M^i = 2\delta^{ij} I \tag{1}$$

We can find the possible eigenvalues as follows. Suppose we choose an orthonormal basis (such a basis always exists for a hermitian matrix) $\{e\}$in which M^i is diagonal for one particular value of i. That is, for a basis vector $|e_k\rangle$ in this basis, we have $M^i |e_k\rangle = \omega_k^i |e_k\rangle$, where ω_k^i is the kth eigenvalue of M^i.

Then with $i = j$ above, we have

$$2\left(M^i\right)^2 = 2I \tag{2}$$

$$\left(M^i\right)^2 = I \tag{3}$$

Operating on a vector e from this basis, we get

$$\left(M^i\right)^2 |e_k\rangle = |e_k\rangle \tag{4}$$

$$= \left(\omega_k^i\right)^2 |e_k\rangle \tag{5}$$

Therefore, the possible values of ω_k^i are ± 1. We didn't choose any particular value for i, so this is true of all four matrices.

Now, for $i \neq j$ we have

$$M^i M^j = -M^j M^i \tag{6}$$

We can find the trace of M^j as follows. Assuming $i \neq j$

$$\mathrm{Tr}M^j = \mathrm{Tr}\left(M^i M^i M^j\right) \tag{7}$$
$$= -\mathrm{Tr}\left(M^i M^j M^i\right) \tag{8}$$
$$= -\mathrm{Tr}\left(M^i M^i M^j\right) \tag{9}$$
$$= -\mathrm{Tr}M^j \tag{10}$$

In line 1 we used 3, in line 2 we used 6 and in line 3 we used the cyclic property of the trace. Thus $\mathrm{Tr}M^j = -\mathrm{Tr}M^j = 0$.

Since each M^j has zero trace, the trace is the sum of the eigenvalues and the possible eigenvalues are ±1, the eigenvalue $+1$ must occur the same number of times as -1, meaning that each M^j must have an even number of eigenvalues, so the matrices must be even-dimensional.

PINGBACKS

Pingback: Dirac equation: derivation