

ANGULAR MOMENTUM AS AN EIGENVECTOR PROBLEM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.9.

The angular momentum in classical mechanics of a collection of point masses m_a located at positions \mathbf{r}_a and moving with a common angular velocity $\boldsymbol{\omega}$ about a common axis is given by

$$\mathbf{L} = \sum_a m_a (\mathbf{r}_a \times \mathbf{v}_a) \quad (1)$$

where $\mathbf{v}_a = \boldsymbol{\omega} \times \mathbf{r}_a$ is the linear velocity of m_a . We can use the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

to write

$$\mathbf{r}_a \times \mathbf{v}_a = \mathbf{r}_a \times (\boldsymbol{\omega} \times \mathbf{r}_a) \quad (2)$$

$$= r_a^2 \boldsymbol{\omega} - \mathbf{r}_a (\mathbf{r}_a \cdot \boldsymbol{\omega}) \quad (3)$$

In terms of components, this is

$$[\mathbf{r}_a \times \mathbf{v}_a]_i = r_a^2 \omega_i - (r_a)_i \sum_j (r_a)_j \omega_j \quad (4)$$

$$= \sum_j \left[r_a^2 \omega_j \delta_{ij} - (r_a)_i (r_a)_j \omega_j \right] \quad (5)$$

$$= \sum_j \left[r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j \quad (6)$$

We can therefore write the angular momentum as

$$\mathbf{L}_i = \sum_j \sum_a m_a \left[r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j \quad (7)$$

$$\equiv \sum_j M_{ij} \omega_j \quad (8)$$

where the matrix M is

$$M_{ij} \equiv \sum_a m_a \left[r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \quad (9)$$

From the definition, we see that M is real and symmetric (interchanging i and j shows that $M_{ij} = M_{ji}$), so M is hermitian.

In Dirac's notation, we have the matrix equation

$$|L\rangle = M |\omega\rangle \quad (10)$$

From this equation, we can see that \mathbf{L} and ω are parallel only if ω is an eigenvector of M . If the eigenvalues of M are non-degenerate, there are therefore three directions for ω such that \mathbf{L} and ω are parallel, and these directions can be found by solving for the eigenvectors of M .

If some of the eigenvalues are degenerate, then there is a range of directions over which \mathbf{L} and ω can be parallel. In the case of a sphere, all 3 eigenvalues of M must be the same, as all directions are axes of symmetry of the sphere.

As an example, suppose we have only one mass $m = 1$ with position

$$\mathbf{r} = [1, 1, 0] \quad (11)$$

We can work out M by substituting into 9:

$$M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (12)$$

The eigenvalues are 0, 2 and 2 with corresponding eigenvectors

$$|\lambda = 0\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (13)$$

$$|\lambda = 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad (14)$$

Thus if ω is a linear combination of the two eigenvectors for $\lambda = 2$, it will be parallel to \mathbf{L} . If ω is parallel to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{L} = 0$, as in this case ω is parallel to \mathbf{r} so $\omega \times \mathbf{r} = 0$, and the mass is located on the axis of rotation so has no angular momentum.