

## ANGULAR MOMENTUM AS AN EIGENVECTOR PROBLEM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.9.

The angular momentum in classical mechanics of a collection of point masses  $m_a$  located at positions  $\mathbf{r}_a$  and moving with a common angular velocity  $\boldsymbol{\omega}$  about a common axis is given by

$$(0.1) \quad \mathbf{L} = \sum_a m_a (\mathbf{r}_a \times \mathbf{v}_a)$$

where  $\mathbf{v}_a = \boldsymbol{\omega} \times \mathbf{r}_a$  is the linear velocity of  $m_a$ . We can use the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

to write

$$(0.2) \quad \mathbf{r}_a \times \mathbf{v}_a = \mathbf{r}_a \times (\boldsymbol{\omega} \times \mathbf{r}_a)$$

$$(0.3) \quad = r_a^2 \boldsymbol{\omega} - \mathbf{r}_a (\mathbf{r}_a \cdot \boldsymbol{\omega})$$

In terms of components, this is

$$(0.4) \quad [\mathbf{r}_a \times \mathbf{v}_a]_i = r_a^2 \omega_i - (r_a)_i \sum_j (r_a)_j \omega_j$$

$$(0.5) \quad = \sum_j \left[ r_a^2 \omega_j \delta_{ij} - (r_a)_i (r_a)_j \omega_j \right]$$

$$(0.6) \quad = \sum_j \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j$$

We can therefore write the angular momentum as

$$(0.7) \quad \mathbf{L}_i = \sum_j \sum_a m_a \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j$$

$$(0.8) \quad \equiv \sum_j M_{ij} \omega_j$$

where the matrix  $M$  is

$$(0.9) \quad M_{ij} \equiv \sum_a m_a \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right]$$

From the definition, we see that  $M$  is real and symmetric (interchanging  $i$  and  $j$  shows that  $M_{ij} = M_{ji}$ ), so  $M$  is hermitian.

In Dirac's notation, we have the matrix equation

$$(0.10) \quad |L\rangle = M|\omega\rangle$$

From this equation, we can see that  $\mathbf{L}$  and  $\omega$  are parallel only if  $\omega$  is an eigenvector of  $M$ . If the eigenvalues of  $M$  are non-degenerate, there are therefore three directions for  $\omega$  such that  $\mathbf{L}$  and  $\omega$  are parallel, and these directions can be found by solving for the eigenvectors of  $M$ .

If some of the eigenvalues are degenerate, then there is a range of directions over which  $\mathbf{L}$  and  $\omega$  can be parallel. In the case of a sphere, all 3 eigenvalues of  $M$  must be the same, as all directions are axes of symmetry of the sphere.

As an example, suppose we have only one mass  $m = 1$  with position

$$(0.11) \quad \mathbf{r} = [1, 1, 0]$$

We can work out  $M$  by substituting into 0.9:

$$(0.12) \quad M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The eigenvalues are 0, 2 and 2 with corresponding eigenvectors

$$(0.13) \quad |\lambda = 0\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(0.14) \quad |\lambda = 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus if  $\omega$  is a linear combination of the two eigenvectors for  $\lambda = 2$ , it will be parallel to  $\mathbf{L}$ . If  $\omega$  is parallel to  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{L} = 0$ , as in this case  $\omega$  is parallel to  $\mathbf{r}$  so  $\omega \times \mathbf{r} = 0$ , and the mass is located on the axis of rotation so has no angular momentum.