COUPLED MASSES ON SPRINGS - A SOLUTION USING MATRIX DIAGONALIZATION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.11.

Here's a practical example of how changing the basis by diagonalizing a hermitian matrix can make a problem easier to solve. Suppose we have two identical masses m free to slide in one dimension on a frictionless horizontal surface. The two masses are connected to 3 springs, with the spring on the left attached to a solid support at one end and to mass #1 at the other, the middle spring connected between the two masses, and the spring on the right connected to mass #2 at one end and to a solid support at the other. The springs all have spring constant k. Define two coordinates x_1 and x_2 to be the positions of the two masses, with $x_i = 0$ corresponding to the location at which mass i is at rest in equilibrium.

Now suppose that the two masses are displaced from their respective equilibrium points, so that x_1 and x_2 are non-zero. The length of the spring to the left of mass 1 is changed (stretched or compressed, depending on the sign of x_1) by x_1, so exerts a force $F_1 = -kx_1$ on mass 1. The length of the spring in the middle is changed by $x_2 - x_1$, so it exerts a force $F_{12} = k(x_2 - x_1)$ on mass 1, and an equal and opposite force $F_{21} = -k(x_2 - x_1)$ on mass 2. Finally, the length of the spring on the right is changed by x_2 and exerts a force $F_2 = -kx_2$ on mass 2. By applying Newton's law $F = ma$, we get the set of equations of motion:

$$\ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2 \tag{1}$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2 \tag{2}$$

While it's possible to solve such a coupled system directly, we can see how an easier method can be found by using matrix algebra. The 2 equations above can be written as a matrix equation

$$|\ddot{x}(t)\rangle = \Omega|x(t)\rangle \tag{3}$$

If we use the basis in which the displacement of each mass is taken to be independent of the other, we have the two basis vectors

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

$$|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

In this basis

$$|x(t)\rangle = x_1(t)|1\rangle + x_2(t)|2\rangle \tag{6}$$

Here, the x_is are just numbers; the vector nature of the equation is delegated to the basis vectors.

In this basis, Ω is the operator whose matrix form is

$$\Omega = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{bmatrix} \tag{7}$$

Since Ω is hermitian, it can be diagonalized by finding its eigenvalues and normalized eigenvectors, and forming a unitary operator U whose columns are these eigenvectors. The basis vectors are now these eigenvectors $|I\rangle$ and $|II\rangle$ (I'm sticking to Shankar's notation, even though it's a bit clumsy), and they are found from $|1\rangle$ and $|2\rangle$ by applying the unitary transformation, that is

$$|I\rangle = U|1\rangle \tag{8}$$

$$|II\rangle = U|2\rangle \tag{9}$$

These transformations can be inverted:

$$|1\rangle = U^\dagger|I\rangle \tag{10}$$

$$|2\rangle = U^\dagger|II\rangle \tag{11}$$

Thus we can insert this into 3 and use $UU^\dagger = I$ to get

$$U^\dagger|\ddot{x}(t)\rangle = U^\dagger\Omega UU^\dagger|x(t)\rangle \tag{12}$$

and $U^\dagger\Omega U$ is the diagonalized version of Ω.

Shankar goes through the details of the calculation, with the results

$$|I\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{13}$$

$$|II\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{14}$$

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{15}$$

$$U^{\dagger}\Omega U = \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{bmatrix} \tag{16}$$

where

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{17}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} \tag{18}$$

Using $|I\rangle$ and $|II\rangle$ as the basis, the differential equations become decoupled, and we have

$$\ddot{x}_i + \omega_i^2 x_i = 0 \tag{19}$$

for $i = I, II$.

Second order ODEs require two initial conditions to be fully solved, and here we're assuming that both masses start off at rest, so that $\dot{x}_i(t) = 0$ for $i = I, II$. In this case, the solutions are

$$x_i(t) = x_i(0)\cos\omega_i t \tag{20}$$

for $i = I, II$.

(A full, general solution would also have a $\sin\omega_i t$ term, but this disappears because we require $\dot{x}_i(t) = 0$.)

The vector solution in the diagonal basis is therefore

$$\begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = |I\rangle x_I(0)\cos\omega_I t + |II\rangle x_{II}(0)\cos\omega_{II} t \tag{21}$$

We now need to figure out what the coefficients $x_I(0)$ and $x_{II}(0)$ are. Assuming we know the initial position of each mass in the original basis as $x_1(0)$ and $x_2(0)$, we can find $x_I(0)$ and $x_{II}(0)$ by projecting $x_1(0)$ and $x_2(0)$ onto the basis $|I\rangle$ and $|II\rangle$. That is, we have

$$\begin{bmatrix} x_I(0) \\ x_{II}(0) \end{bmatrix} = U \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{22}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{23}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} x_1(0) + x_2(0) \\ x_1(0) - x_2(0) \end{bmatrix} \tag{24}$$

We get

(25)

$$\begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = \frac{x_1(0) + x_2(0)}{\sqrt{2}} |I\rangle \cos\omega_I t + \frac{x_1(0) - x_2(0)}{\sqrt{2}} |II\rangle \cos\omega_{II} t$$

(26)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} [x_1(0) + x_2(0)] \cos\sqrt{\frac{k}{m}}t + [x_1(0) - x_2(0)] \cos\sqrt{\frac{3k}{m}}t \\ [x_1(0) + x_2(0)] \cos\sqrt{\frac{k}{m}}t - [x_1(0) - x_2(0)] \cos\sqrt{\frac{3k}{m}}t \end{bmatrix}$$

where in the last line we substituted using 13 to write everything in terms of the original basis $|1\rangle$ and $|2\rangle$.

For the special case where the initial positions are given by $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $x_1(0) = 1$ and $x_2(0) = 0$, so that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \end{bmatrix} \tag{27}$$

Going back to 26, we can write the solution as a matrix equation

(28)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

The matrix with the cosines is independent of the initial state, so that once we know this matrix, we can work out the general solution as a function of time for any initial state. The matrix is known as the *propagator*. [Although Shankar uses the symbol $U(t)$ to refer to the propagator, it's *not* a unitary matrix. For example, its determinant is $\cos\left(\sqrt{\frac{k}{m}}t\right)\cos\left(\sqrt{\frac{3k}{m}}t\right) \neq 1$ for $t \neq 0$.]

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