

## EXPONENTIALS OF OPERATORS - BAKER-CAMPBELL-HAUSDORFF FORMULA

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.9.

Although the result in this post isn't covered in Shankar's book, it's a result that is frequently used in quantum theory, so it's worth including at this point.

We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$(1) \quad f(\Omega) = e^{i\Omega}$$

If  $\Omega$  is hermitian, the exponential  $e^{i\Omega}$  is unitary. If we try to calculate the exponential of two operators such as  $e^{A+B}$ , the result isn't as simple as we might hope if  $A$  and  $B$  don't commute. To see the problem, we can write this out as a power series

$$(2) \quad e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$

$$(3) \quad = I + A + B + \frac{1}{2}(A+B)(A+B) + \dots$$

$$(4) \quad = I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots$$

The problem appears first in the fourth term in the series, since we can't condense the  $AB + BA$  sum into  $2AB$  if  $[A, B] \neq 0$ . In fact, the expansion of  $e^A e^B$  can be written entirely in terms of the commutators of  $A$  and  $B$  with each other, nested to increasingly higher levels. This formula is known as the Baker-Campbell-Hausdorff formula. Up to the fourth order commutator, the BCH formula gives

$$(5) \quad e^A e^B = \exp \left[ A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) - \frac{1}{24}[B, [A, [A, B]]] + \dots \right]$$

There is no known closed form expression for this result. However, an important special case that occurs frequently in quantum theory is the case

where  $[A, B] = cI$ , where  $c$  is a complex scalar and  $I$  is the usual identity matrix. Since  $cI$  commutes with all operators, all terms from the third order upwards are zero, and we have

$$(6) \quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

We can prove this result as follows. Start with the operator function

$$(7) \quad G(t) \equiv e^{t(A+B)} e^{-tA}$$

where  $t$  is a scalar parameter (not necessarily time!).

From its definition,

$$(8) \quad G(0) = I$$

The inverse is

$$(9) \quad G^{-1}(t) = e^{tA} e^{-t(A+B)}$$

and the derivative is

$$(10) \quad \frac{dG(t)}{dt} = (A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A$$

Note that we have to keep the  $(A+B)$  factor to the left of the  $A$  factor because  $[A, B] \neq 0$ . Now we multiply:

$$(11) \quad G^{-1} \frac{dG}{dt} = e^{tA} e^{-t(A+B)} \left[ (A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A \right]$$

$$(12) \quad = e^{tA} (A+B) e^{-tA} - A$$

$$(13) \quad = e^{tA} A e^{-tA} + e^{tA} B e^{-tA} - A$$

$$(14) \quad = e^{tA} B e^{-tA}$$

$$(15) \quad = B + t[A, B]$$

$$(16) \quad = B + ctI$$

We used Hadamard's lemma in the penultimate line, which in this case reduces to

$$(17) \quad e^{tA} B e^{-tA} = B + t[A, B]$$

because  $[A, B] = cI$  so all higher order commutators are zero.

We end up with an expression in which  $A$  has disappeared. This gives the differential equation for  $G$ :

$$(18) \quad G^{-1} \frac{dG}{dt} = B + ctI$$

We try a solution of the form (this apparently appears from divine inspiration):

$$(19) \quad G(t) = e^{\alpha t B} e^{\beta c t^2}$$

From which we get

$$(20) \quad G^{-1} = e^{-\alpha t B} e^{-\beta c t^2}$$

$$(21) \quad \frac{dG}{dt} = (\alpha B + 2\beta c t) e^{\alpha t B} e^{\beta c t^2}$$

$$(22) \quad G^{-1} \frac{dG}{dt} = \alpha B + 2\beta c t$$

Comparing this to 18, we have

$$(23) \quad \alpha = 1$$

$$(24) \quad \beta = \frac{1}{2}$$

$$(25) \quad G(t) = e^{tB} e^{\frac{1}{2} c t^2}$$

Setting this equal to the original definition of  $G$  in 7 and then taking  $t = 1$  we have

$$(26) \quad e^{A+B} e^{-A} = e^B e^{c/2}$$

$$(27) \quad e^{A+B} = e^B e^A e^{\frac{1}{2} c}$$

$$(28) \quad = e^B e^A e^{\frac{1}{2} [A, B]}$$

If we swap  $A$  with  $B$  and use the fact that  $A + B = B + A$ , and also  $[A, B] = -[B, A]$ , we have

$$(29) \quad e^{A+B} = e^A e^B e^{-\frac{1}{2} [A, B]}$$

This is the restricted form of the BCH formula for the case where  $[A, B]$  is a scalar.

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Pingback: Time-dependent propagators

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Pingback: Linear chain of oscillators - External force, unitary operator

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