EXPONENTIALS OF OPERATORS - HADAMARD'S LEMMA

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.9.

Although the result in this post isn't covered in Shankar's book, it's a result that is frequently used in quantum theory, so it's worth including at this point.

We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega)=e^{i\Omega} \tag{1}$$

Here we'll look at a special function of two operators of the form

$$h(A,B)=e^{A}Be^{-A} \tag{2}$$

If $[A,B]=0$, we can cancel the two exponentials and get the result $h(A,B)=B$. However, if $[A,B]\neq 0$ the two exponentials must remain separated by the middle B operator. To get a simpler form for this function, we'll consider the auxiliary function

$$f(t)=e^{tA}Be^{-tA} \tag{3}$$

where t is some parameter. We'll need the first 3 derivatives at $t=0$:

$$\begin{aligned}
f(0) &= B && (4)\\
f'(t) &= Ae^{tA}Be^{-tA}-e^{tA}Be^{-tA}A && (5)\\
&= e^{tA}[A,B]e^{-tA} && (6)\\
f'(0) &= [A,B] && (7)\\
f''(t) &= Ae^{tA}[A,B]e^{-tA}-e^{tA}[A,B]e^{-tA}A && (8)\\
&= e^{tA}[A,[A,B]]e^{-tA} && (9)\\
f''(0) &= [A,[A,B]] && (10)\\
f'''(t) &= e^{tA}[A,[A,[A,B]]]e^{-tA} && (11)\\
f'''(0) &= [A,[A,[A,B]]] && (12)
\end{aligned}$$

We can now write a Taylor expansion of 3 around $t=0$:

$$e^{tA}Be^{-tA} = f(0) + tf'(0) + \frac{t^2}{2}f''(0) + \frac{t^3}{6}f'''(0) + \ldots \tag{13}$$

$$= B + [A,B]t + [A,[A,B]]\frac{t^2}{2!} + [A,[A,[A,B]]]\frac{t^3}{3!} + \ldots \tag{14}$$

Taking $t = 1$ gives the required expansion

$$e^A Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots \tag{15}$$

This is known as *Hadamard's lemma*.
If we introduce the notation

$$\mathrm{ad}_A(B) \equiv [A,B] \tag{16}$$

$$\mathrm{ad}_A\mathrm{ad}_A(B) = [A,[A,B]] \tag{17}$$

and in general $(\mathrm{ad}_A)^n(B)$ is the nth order commutator of A with B, then we can write Hadamard's lemma as

$$e^A Be^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{ad}_A)^n(B) \tag{18}$$

$$= \exp(\mathrm{ad}_A)(B) \tag{19}$$

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