

EXPONENTIALS OF OPERATORS - HADAMARD'S LEMMA

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.9.

Although the result in this post isn't covered in Shankar's book, it's a result that is frequently used in quantum theory, so it's worth including at this point.

We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$(0.1) \quad f(\Omega) = e^{i\Omega}$$

Here we'll look at a special function of two operators of the form

$$(0.2) \quad h(A, B) = e^A B e^{-A}$$

If $[A, B] = 0$, we can cancel the two exponentials and get the result $h(A, B) = B$. However, if $[A, B] \neq 0$ the two exponentials must remain separated by the middle B operator. To get a simpler form for this function, we'll consider the auxiliary function

$$(0.3) \quad f(t) = e^{tA} B e^{-tA}$$

where t is some parameter. We'll need the first 3 derivatives at $t = 0$:

$$(0.4) \quad f(0) = B$$

$$(0.5) \quad f'(t) = A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A$$

$$(0.6) \quad = e^{tA} [A, B] e^{-tA}$$

$$(0.7) \quad f'(0) = [A, B]$$

$$(0.8) \quad f''(t) = A e^{tA} [A, B] e^{-tA} - e^{tA} [A, B] e^{-tA} A$$

$$(0.9) \quad = e^{tA} [A, [A, B]] e^{-tA}$$

$$(0.10) \quad f''(0) = [A, [A, B]]$$

$$(0.11) \quad f'''(t) = e^{tA} [A, [A, [A, B]]] e^{-tA}$$

$$(0.12) \quad f'''(0) = [A, [A, [A, B]]]$$

We can now write a Taylor expansion of 0.3 around $t = 0$:

$$(0.13) \quad e^{tA} B e^{-tA} = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(0) + \dots$$

$$(0.14) \quad = B + [A, B]t + [A, [A, B]] \frac{t^2}{2!} + [A, [A, [A, B]]] \frac{t^3}{3!} + \dots$$

Taking $t = 1$ gives the required expansion

$$(0.15) \quad e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots$$

This is known as *Hadamard's lemma*.

If we introduce the notation

$$(0.16) \quad \text{ad}_A(B) \equiv [A, B]$$

$$(0.17) \quad \text{ad}_A \text{ad}_A(B) = [A, [A, B]]$$

and in general $(\text{ad}_A)^n(B)$ is the n th order commutator of A with B , then we can write Hadamard's lemma as

$$(0.18) \quad e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_A)^n(B)$$

$$(0.19) \quad = \exp(\text{ad}_A)(B)$$

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