DIFFERENTIAL OPERATOR - EIGENVALUES AND EIGENSTATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10.

Continuing with our study of differential operators, we'll look now at their eigenvalues and eigenstates. The operator we're studying is

$$K=-i\frac{d}{dx} \tag{1}$$

The eigenvalue equation is as usual:

$$K\left|k\right\rangle =k\left|k\right\rangle \tag{2}$$

where $\left|k\right\rangle$is an eigenstate and k (outside the ket) is a (possibly complex) scalar. To find $\left|k\right\rangle$, we form the matrix element with $\left\langle x\right|$ and insert the unit operator:

$$\begin{aligned}\left\langle x\left|K\right|k\right\rangle &=k\left\langle x\left|k\right.\right\rangle &(3)\\ \left\langle x\left|K\right|k\right\rangle &=\int\left\langle x\left|K\right|x^{\prime}\right\rangle \left\langle x^{\prime}\left|k\right.\right\rangle dx^{\prime} &(4)\\ &=-i\int\delta^{\prime}\left(x-x^{\prime}\right)\psi_{k}\left(x^{\prime}\right)dx^{\prime} &(5)\\ &=-i\frac{d}{dx}\psi_{k}\left(x\right) &(6)\end{aligned}$$

In the third line we used the matrix element

$$\left\langle x\left|K\right|x^{\prime}\right\rangle =-i\delta^{\prime}\left(x-x^{\prime}\right) \tag{7}$$

Equating the RHS on the first and last lines gives the differential equation

$$-i\frac{d}{dx}\psi_{k}\left(x\right)=k\psi_{k}\left(x\right) \tag{8}$$

which has the solution

$$\psi_{k}\left(x\right)=Ae^{ikx} \tag{9}$$

where A is a constant of integration. In order for $\psi_k(x)$ to be bounded as $x \to \pm\infty$, k must be real, so we'll restrict our attention to that case. The usual choice for A is $1/\sqrt{2\pi}$ so that

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \tag{10}$$

This leads to the normalization condition

$$\begin{aligned} \langle k | k' \rangle &= \int_{-\infty}^{\infty} \langle k | x \rangle \langle x | k' \rangle \, dx &(11)\\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k')x} dx &(12)\\ &= \delta\left(k - k'\right) &(13) \end{aligned}$$

where in the last line we used the traditional formula for the delta function. Thus the $|k\rangle$ basis is orthogonal, and normalized the same way as the $|x\rangle$ basis.

To convert between the $|k\rangle$ and $|x\rangle$ bases, we can use the unit operator in the two bases. Thus for some vector (function) $|f\rangle$ we have

$$f(k) = \langle k | f \rangle = \int \langle k | x \rangle \langle x | f \rangle \, dx = \int \psi_k^*(x) f(x) \, dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x) \tag{14}$$

Thus $f(k)$ is the Fourier transform of $f(x)$. We can use the same procedure to go in the reverse direction:

$$f(x) = \langle x | f \rangle = \int \langle x | k \rangle \langle k | f \rangle \, dk = \int \psi_k(x) f(k) \, dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(k) \tag{15}$$

The effect of the position operator X on a vector $|f(x)\rangle$ can be found by inserting the unit operator:

$$\begin{aligned} \langle x | X | f \rangle &= \int \left\langle x | X | x' \right\rangle \left\langle x' | f \right\rangle dx' &(16)\\ &= \int x' \left\langle x | x' \right\rangle \left\langle x' | f \right\rangle dx' &(17)\\ &= \int x' \delta\left(x - x'\right) \left\langle x' | f \right\rangle dx' &(18)\\ &= x \langle x | f \rangle &(19) \end{aligned}$$

Thus X just multiplies any function of x by x itself. A similar argument in the $|k\rangle$ basis shows that

$$\langle k|K|f(k)\rangle = k\langle k|f(k)\rangle \tag{20}$$

We can use similar calculations to find the matrix elements of K in the $|x\rangle$ basis and of X (the position operator) in the $|k\rangle$ basis. We get

$$\left\langle k|X|k'\right\rangle = \int\int \langle k|x\rangle \left\langle x|X|x'\right\rangle \left\langle x'|k'\right\rangle dx\,dx' \tag{21}$$

$$= \frac{1}{2\pi}\int\int e^{-ikx}x'\left\langle x|x'\right\rangle e^{ik'x'}dx\,dx' \tag{22}$$

$$= \frac{1}{2\pi}\int\int e^{-ikx}x'\delta\left(x-x'\right)e^{ik'x'}dx\,dx' \tag{23}$$

$$= \frac{1}{2\pi}\int xe^{i(k'-k)x}dx \tag{24}$$

$$= i\frac{d}{dk}\left[\frac{1}{2\pi}\int e^{i(k'-k)x}dx\right] \tag{25}$$

$$= i\delta'\left(k-k'\right) \tag{26}$$

The action of X on an arbitrary vector $|g\rangle$ in the k basis can be found from this:

$$\langle k|X|g(k)\rangle = \int \left\langle k|X|k'\right\rangle \left\langle k'|g\right\rangle dk' \tag{27}$$

$$= i\int \delta'\left(k-k'\right)g\left(k'\right)dk' \tag{28}$$

$$= i\frac{dg(k)}{dk} \tag{29}$$

$$= i\left\langle k\left|\frac{dg(k)}{dk}\right.\right\rangle \tag{30}$$

where in the third line we've used the property of $\delta'(k-k')$ mentioned here.

By a similar calculation, we can find the matrix elements of K in the $|x\rangle$ basis:

$$\left\langle x\left|K\right|x'\right\rangle = \int\int\left\langle x\left|k\right.\right\rangle\left\langle k\left|K\right|k'\right\rangle\left\langle k'\left|x'\right.\right\rangle dk\,dk' \tag{31}$$

$$= \frac{1}{2\pi}\int\int e^{ikx}k'\left\langle k\left|k'\right.\right\rangle e^{-ik'x'}dk\,dk' \tag{32}$$

$$= \frac{1}{2\pi}\int\int e^{ikx}k'\delta\left(k-k'\right)e^{-ik'x'}dk\,dk' \tag{33}$$

$$= \frac{1}{2\pi}\int xe^{i(x-x')k}dk \tag{34}$$

$$= -i\frac{d}{dx}\left[\frac{1}{2\pi}\int e^{i(x-x')k}dk\right] \tag{35}$$

$$= -i\delta'\left(x-x'\right) \tag{36}$$

Similarly, we have

$$\left\langle x\left|K\right|g\left(x\right)\right\rangle = \int\left\langle x\left|K\right|x'\right\rangle\left\langle x'\left|g\right.\right\rangle dx' \tag{37}$$

$$= -i\int\delta'\left(x-x'\right)g\left(x'\right)dx' \tag{38}$$

$$= -i\frac{dg\left(x\right)}{dx} \tag{39}$$

$$= -i\left\langle x\left|\frac{dg\left(x\right)}{dx}\right.\right\rangle \tag{40}$$

From 30 and 40 we can work out the familiar commutator. Just for variety, we'll do this in the $\left|k\right\rangle$ basis:

$$XK\left|f\left(k\right)\right\rangle = X\left[k\left|f\left(k\right)\right\rangle\right] \tag{41}$$

$$= i\frac{d}{dk}\left[k\left|f\left(k\right)\right\rangle\right] \tag{42}$$

$$= i\left[\left|f\left(k\right)\right\rangle + k\left|\frac{df}{dk}\right\rangle\right] \tag{43}$$

$$KX\left|f\left(k\right)\right\rangle = iK\left|\frac{df}{dk}\right\rangle \tag{44}$$

$$= ik\left|\frac{df}{dk}\right\rangle \tag{45}$$

Therefore

$$\left[X,K\right]\left|f\left(k\right)\right\rangle = i\left|f\left(k\right)\right\rangle \tag{46}$$

or, looking just at the operators

$$[X,K]=iI \tag{47}$$

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