

DIFFERENTIAL OPERATOR - EIGENVALUES AND EIGENSTATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10.

Continuing with our study of differential operators, we'll look now at their eigenvalues and eigenstates. The operator we're studying is

$$K = -i \frac{d}{dx} \quad (1)$$

The eigenvalue equation is as usual:

$$K |k\rangle = k |k\rangle \quad (2)$$

where $|k\rangle$ is an eigenstate and k (outside the ket) is a (possibly complex) scalar. To find $|k\rangle$, we form the matrix element with $\langle x|$ and insert the unit operator:

$$\langle x | K | k \rangle = k \langle x | k \rangle \quad (3)$$

$$\langle x | K | k \rangle = \int \langle x | K | x' \rangle \langle x' | k \rangle dx' \quad (4)$$

$$= -i \int \delta'(x-x') \psi_k(x') dx' \quad (5)$$

$$= -i \frac{d}{dx} \psi_k(x) \quad (6)$$

In the third line we used the matrix element

$$\langle x | K | x' \rangle = -i \delta'(x-x') \quad (7)$$

Equating the RHS on the first and last lines gives the differential equation

$$-i \frac{d}{dx} \psi_k(x) = k \psi_k(x) \quad (8)$$

which has the solution

$$\psi_k(x) = A e^{ikx} \quad (9)$$

where A is a constant of integration. In order for $\psi_k(x)$ to be bounded as $x \rightarrow \pm\infty$, k must be real, so we'll restrict our attention to that case. The usual choice for A is $1/\sqrt{2\pi}$ so that

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \quad (10)$$

This leads to the normalization condition

$$\langle k|k' \rangle = \int_{-\infty}^{\infty} \langle k|x \rangle \langle x|k' \rangle dx \quad (11)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k')x} dx \quad (12)$$

$$= \delta(k-k') \quad (13)$$

where in the last line we used the traditional formula for the delta function. Thus the $|k\rangle$ basis is orthogonal, and normalized the same way as the $|x\rangle$ basis.

To convert between the $|k\rangle$ and $|x\rangle$ bases, we can use the unit operator in the two bases. Thus for some vector (function) $|f\rangle$ we have

$$f(k) = \langle k|f \rangle = \int \langle k|x \rangle \langle x|f \rangle dx = \int \psi_k^*(x) f(x) dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x) dx \quad (14)$$

Thus $f(k)$ is the Fourier transform of $f(x)$. We can use the same procedure to go in the reverse direction:

$$f(x) = \langle x|f \rangle = \int \langle x|k \rangle \langle k|f \rangle dk = \int \psi_k(x) f(k) dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(k) dk \quad (15)$$

The effect of the position operator X on a vector $|f(x)\rangle$ can be found by inserting the unit operator:

$$\langle x|X|f \rangle = \int \langle x|X|x' \rangle \langle x'|f \rangle dx' \quad (16)$$

$$= \int x' \langle x|x' \rangle \langle x'|f \rangle dx' \quad (17)$$

$$= \int x' \delta(x-x') \langle x'|f \rangle dx' \quad (18)$$

$$= x \langle x|f \rangle \quad (19)$$

Thus X just multiplies any function of x by x itself. A similar argument in the $|k\rangle$ basis shows that

$$\langle k|K|f(k)\rangle = k\langle k|f(k)\rangle \quad (20)$$

We can use similar calculations to find the matrix elements of K in the $|x\rangle$ basis and of X (the position operator) in the $|k\rangle$ basis. We get

$$\langle k|X|k'\rangle = \int \int \langle k|x\rangle \langle x|X|x'\rangle \langle x'|k'\rangle dx dx' \quad (21)$$

$$= \frac{1}{2\pi} \int \int e^{-ikx} x' \langle x|x'\rangle e^{ik'x'} dx dx' \quad (22)$$

$$= \frac{1}{2\pi} \int \int e^{-ikx} x' \delta(x-x') e^{ik'x'} dx dx' \quad (23)$$

$$= \frac{1}{2\pi} \int x e^{i(k'-k)x} dx \quad (24)$$

$$= i \frac{d}{dk} \left[\frac{1}{2\pi} \int e^{i(k'-k)x} dx \right] \quad (25)$$

$$= i\delta'(k-k') \quad (26)$$

The action of X on an arbitrary vector $|g\rangle$ in the k basis can be found from this:

$$\langle k|X|g(k)\rangle = \int \langle k|X|k'\rangle \langle k'|g\rangle dk' \quad (27)$$

$$= i \int \delta'(k-k') g(k') dk' \quad (28)$$

$$= i \frac{dg(k)}{dk} \quad (29)$$

$$= i \left\langle k \left| \frac{dg(k)}{dk} \right. \right\rangle \quad (30)$$

where in the third line we've used the property of $\delta'(k-k')$ mentioned here.

By a similar calculation, we can find the matrix elements of K in the $|x\rangle$ basis:

$$\langle x|K|x' \rangle = \int \int \langle x|k \rangle \langle k|K|k' \rangle \langle k'|x' \rangle dk dk' \quad (31)$$

$$= \frac{1}{2\pi} \int \int e^{ikx} k' \langle k|k' \rangle e^{-ik'x'} dk dk' \quad (32)$$

$$= \frac{1}{2\pi} \int \int e^{ikx} k' \delta(k-k') e^{-ik'x'} dk dk' \quad (33)$$

$$= \frac{1}{2\pi} \int x e^{i(x-x')k} dk \quad (34)$$

$$= -i \frac{d}{dx} \left[\frac{1}{2\pi} \int e^{i(x-x')k} dk \right] \quad (35)$$

$$= -i \delta'(x-x') \quad (36)$$

Similarly, we have

$$\langle x|K|g(x) \rangle = \int \langle x|K|x' \rangle \langle x'|g \rangle dx' \quad (37)$$

$$= -i \int \delta'(x-x') g(x') dx' \quad (38)$$

$$= -i \frac{dg(x)}{dx} \quad (39)$$

$$= -i \left\langle x \left| \frac{dg(x)}{dx} \right. \right\rangle \quad (40)$$

From 30 and 40 we can work out the familiar commutator. Just for variety, we'll do this in the $|k\rangle$ basis:

$$XK|f(k)\rangle = X[k|f(k)\rangle] \quad (41)$$

$$= i \frac{d}{dk} [k|f(k)\rangle] \quad (42)$$

$$= i \left[|f(k)\rangle + k \left| \frac{df}{dk} \right. \right] \quad (43)$$

$$KX|f(k)\rangle = iK \left| \frac{df}{dk} \right. \rangle \quad (44)$$

$$= ik \left| \frac{df}{dk} \right. \rangle \quad (45)$$

Therefore

$$[X, K]|f(k)\rangle = i|f(k)\rangle \quad (46)$$

or, looking just at the operators

$$[X, K] = iI \quad (47)$$

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