

## DIFFERENTIAL OPERATOR - EIGENVALUES AND EIGENSTATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10.

Continuing with our study of differential operators, we'll look now at their eigenvalues and eigenstates. The operator we're studying is

$$(1) \quad K = -i \frac{d}{dx}$$

The eigenvalue equation is as usual:

$$(2) \quad K |k\rangle = k |k\rangle$$

where  $|k\rangle$  is an eigenstate and  $k$  (outside the ket) is a (possibly complex) scalar. To find  $|k\rangle$ , we form the matrix element with  $\langle x|$  and insert the unit operator:

$$(3) \quad \langle x | K | k \rangle = k \langle x | k \rangle$$

$$(4) \quad \langle x | K | k \rangle = \int \langle x | K | x' \rangle \langle x' | k \rangle dx'$$

$$(5) \quad = -i \int \delta'(x-x') \psi_k(x') dx'$$

$$(6) \quad = -i \frac{d}{dx} \psi_k(x)$$

In the third line we used the matrix element

$$(7) \quad \langle x | K | x' \rangle = -i \delta'(x-x')$$

Equating the RHS on the first and last lines gives the differential equation

$$(8) \quad -i \frac{d}{dx} \psi_k(x) = k \psi_k(x)$$

which has the solution

$$(9) \quad \psi_k(x) = Ae^{ikx}$$

where  $A$  is a constant of integration. In order for  $\psi_k(x)$  to be bounded as  $x \rightarrow \pm\infty$ ,  $k$  must be real, so we'll restrict our attention to that case. The usual choice for  $A$  is  $1/\sqrt{2\pi}$  so that

$$(10) \quad \psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$$

This leads to the normalization condition

$$(11) \quad \langle k|k' \rangle = \int_{-\infty}^{\infty} \langle k|x \rangle \langle x|k' \rangle dx$$

$$(12) \quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k')x} dx$$

$$(13) \quad = \delta(k-k')$$

where in the last line we used the traditional formula for the delta function. Thus the  $|k\rangle$  basis is orthogonal, and normalized the same way as the  $|x\rangle$  basis.

To convert between the  $|k\rangle$  and  $|x\rangle$  bases, we can use the unit operator in the two bases. Thus for some vector (function)  $|f\rangle$  we have

$$(14) \quad f(k) = \langle k|f \rangle = \int \langle k|x \rangle \langle x|f \rangle dx = \int \psi_k^*(x) f(x) dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x)$$

Thus  $f(k)$  is the Fourier transform of  $f(x)$ . We can use the same procedure to go in the reverse direction:

$$(15) \quad f(x) = \langle x|f \rangle = \int \langle x|k \rangle \langle k|f \rangle dk = \int \psi_k(x) f(k) dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(k)$$

The effect of the position operator  $X$  on a vector  $|f(x)\rangle$  can be found by inserting the unit operator:

$$(16) \quad \langle x|X|f\rangle = \int \langle x|X|x'\rangle \langle x'|f\rangle dx'$$

$$(17) \quad = \int x' \langle x|x'\rangle \langle x'|f\rangle dx'$$

$$(18) \quad = \int x' \delta(x-x') \langle x'|f\rangle dx'$$

$$(19) \quad = x \langle x|f\rangle$$

Thus  $X$  just multiplies any function of  $x$  by  $x$  itself. A similar argument in the  $|k\rangle$  basis shows that

$$(20) \quad \langle k|K|f(k)\rangle = k \langle k|f(k)\rangle$$

We can use similar calculations to find the matrix elements of  $K$  in the  $|x\rangle$  basis and of  $X$  (the position operator) in the  $|k\rangle$  basis. We get

$$(21) \quad \langle k|X|k'\rangle = \int \int \langle k|x\rangle \langle x|X|x'\rangle \langle x'|k'\rangle dx dx'$$

$$(22) \quad = \frac{1}{2\pi} \int \int e^{-ikx} x' \langle x|x'\rangle e^{ik'x'} dx dx'$$

$$(23) \quad = \frac{1}{2\pi} \int \int e^{-ikx} x' \delta(x-x') e^{ik'x'} dx dx'$$

$$(24) \quad = \frac{1}{2\pi} \int x e^{i(k'-k)x} dx$$

$$(25) \quad = i \frac{d}{dk} \left[ \frac{1}{2\pi} \int e^{i(k'-k)x} dx \right]$$

$$(26) \quad = i \delta'(k-k')$$

The action of  $X$  on an arbitrary vector  $|g\rangle$  in the  $k$  basis can be found from this:

$$(27) \quad \langle k|X|g(k)\rangle = \int \langle k|X|k'\rangle \langle k'|g\rangle dk'$$

$$(28) \quad = i \int \delta'(k-k') g(k') dk'$$

$$(29) \quad = i \frac{dg(k)}{dk}$$

$$(30) \quad = i \left\langle k \left| \frac{dg(k)}{dk} \right. \right\rangle$$

where in the third line we've used the property of  $\delta'(k - k')$  mentioned here.

By a similar calculation, we can find the matrix elements of  $K$  in the  $|x\rangle$  basis:

$$(31) \quad \langle x|K|x'\rangle = \int \int \langle x|k\rangle \langle k|K|k'\rangle \langle k'|x'\rangle dk dk'$$

$$(32) \quad = \frac{1}{2\pi} \int \int e^{ikx} k' \langle k|k'\rangle e^{-ik'x'} dk dk'$$

$$(33) \quad = \frac{1}{2\pi} \int \int e^{ikx} k' \delta(k - k') e^{-ik'x'} dk dk'$$

$$(34) \quad = \frac{1}{2\pi} \int x e^{i(x-x')k} dk$$

$$(35) \quad = -i \frac{d}{dx} \left[ \frac{1}{2\pi} \int e^{i(x-x')k} dk \right]$$

$$(36) \quad = -i \delta'(x - x')$$

Similarly, we have

$$(37) \quad \langle x|K|g(x)\rangle = \int \langle x|K|x'\rangle \langle x'|g\rangle dx'$$

$$(38) \quad = -i \int \delta'(x - x') g(x') dx'$$

$$(39) \quad = -i \frac{dg(x)}{dx}$$

$$(40) \quad = -i \left\langle x \left| \frac{dg(x)}{dx} \right. \right\rangle$$

From 30 and 40 we can work out the familiar commutator. Just for variety, we'll do this in the  $|k\rangle$  basis:

$$(41) \quad XK |f(k)\rangle = X [k |f(k)\rangle]$$

$$(42) \quad = i \frac{d}{dk} [k |f(k)\rangle]$$

$$(43) \quad = i \left[ |f(k)\rangle + k \left| \frac{df}{dk} \right\rangle \right]$$

$$(44) \quad KX |f(k)\rangle = iK \left| \frac{df}{dk} \right\rangle$$

$$(45) \quad = ik \left| \frac{df}{dk} \right\rangle$$

Therefore

$$(46) \quad [X, K] |f(k)\rangle = i |f(k)\rangle$$

or, looking just at the operators

$$(47) \quad [X, K] = iI$$

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