VIBRATING STRING - NORMAL MODE ANALYSIS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.10. Exercise 1.10.4.

I'll run through Shakar's example 1.10.1 on a vibrating string, so we can see an application of the theory of infinite dimensional spaces. Suppose we have a string (for example, a violin string) that is anchored at $x=0$ and $x=L$. If we pluck the string at $t=0$, its future position is governed by the wave equation:

$$\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2} \tag{1}$$

[For simplicity, we're taking the wave speed to be 1, which is why there's no constant in this equation.] We can write this as an operator equation using the $K=-i\frac{\partial}{\partial x}$ operator we introduced last time. Viewing the wave as a vector in the $|x\rangle$ basis, we then have

$$\left|\ddot{\psi}(t)\right\rangle=-K^2\left|\psi(t)\right\rangle \tag{2}$$

The idea is now to look at the RHS of this equation and diagonalize the K^2 operator by finding its eigenvalues and eigenvectors. Working in the $|x\rangle$ basis, we can write the eigenvalue problem as

$$\left\langle x\left|K^2\right|\psi\right\rangle = k^2\left\langle x\left|\psi\right.\right\rangle \tag{3}$$

$$-\frac{d^2\psi(x)}{dx^2}=k^2\psi(x) \tag{4}$$

This has the general solution

$$\psi(x)=A\cos kx+B\sin kx \tag{5}$$

where A and B are constants of integration, to be determined by the boundary conditions. Since the ends of the string are fixed at $\psi(0)=\psi(L)=0$, we must have $A=0$, and we then have

$$B\sin kL=0 \tag{6}$$

In order to avoid a trivial solution where $\psi(x)=0$ everywhere, we have $B\neq 0$, so

$$kL = m\pi \tag{7}$$

for $m = 1, 2, 3, \ldots$. We therefore have the discrete set of solutions

$$\psi_m(x) = B \sin \frac{m\pi x}{L} \tag{8}$$

We can choose $B = \sqrt{\frac{2}{L}}$ to normalize the solution so that

$$\int \psi_m(x)\, \psi_{m'}(x)\, dx = \delta_{mm'} \tag{9}$$

So far, this is the same as the solution to the infinite square well in quantum mechanics, but now we follow a different path, since we need to satisfy the wave equation 1, and not Schrödinger's equation, which is first order in time.

We now have two different orthonormal bases that can be used to represent the states of the string. The $|x\rangle$ basis is continuous, consisting of all real values of x in the interval $[0, L]$. The other basis is also infinite, but it is discrete, as it consists of the possible values of k as given by 7. Since k is determined by the integer m, we'll call this the $|m\rangle$ basis. In the $|x\rangle$ basis, the state $|m\rangle$ is given by 8:

$$\langle x | m \rangle = \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L} \tag{10}$$

The general solution as a function of time is an abstract vector $|\psi(t)\rangle$. We can project this onto the $|x\rangle$ basis, when we would get

$$\langle x | \psi(t) \rangle = \psi(x, t) \tag{11}$$

Or we can project it onto the $|m\rangle$ basis, which gives that component of $|\psi(t)\rangle$ that is composed of a wave with index m. In the $|m\rangle$ basis, the operator K^2 is diagonal, since

$$K^2 \psi_m(x) = -\sqrt{\frac{2}{L}} \frac{d^2}{dx^2} \sin \frac{m\pi x}{L} = \left(\frac{m\pi}{L}\right)^2 \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L} = \left(\frac{m\pi}{L}\right)^2 \psi_m(x) = k^2 \psi_m(x) \tag{12}$$

We can write the projection of $|\psi(t)\rangle$ onto the $|m\rangle$ basis as $\langle m | \psi(t) \rangle$. Going back to 2, we see that each component $\langle m | \psi(t) \rangle$ individually satisfies the differential equation

$$\frac{d^2}{dt^2} \langle m | \psi(t) \rangle = -\left(\frac{m\pi}{L}\right)^2 \langle m | \psi(t) \rangle \tag{13}$$

This is the same equation as 4, except now we're dealing with a time derivative instead of a space derivative. The solution is therefore of the same form:

$$\langle m|\psi(t)\rangle = C\cos kt + D\sin kt \tag{14}$$

To find C and D, we now use the initial conditions at $t=0$. We'll assume that the string is held in some fixed shape and then released at $t=0$, which means that we need to specify this initial shape as $\langle m|\psi(0)\rangle$, and that the initial velocity is zero. The latter condition means that

$$\frac{d}{dt}\langle m|\psi(0)\rangle = -kC\sin k0 + kD\cos k0 = 0 \tag{15}$$

which gives us $D=0$, so we have

$$\langle m|\psi(t)\rangle = \langle m|\psi(0)\rangle\cos kt = \langle m|\psi(0)\rangle\cos\frac{m\pi t}{L} \tag{16}$$

The general solution is therefore found by inserting the unit operator in the form $1=\sum_m |m\rangle\langle m|$:

$$|\psi(t)\rangle = \sum_m |m\rangle\langle m|\psi(t)\rangle \tag{17}$$

$$= \sum_m |m\rangle\cos\frac{m\pi t}{L}\langle m|\psi(0)\rangle \tag{18}$$

This can be written as a propagator $U(t)$ acting on the initial state:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \tag{19}$$

$$U(t) \equiv \sum_m |m\rangle\langle m|\cos\frac{m\pi t}{L} \tag{20}$$

Just as with our earlier example of two masses coupled by springs, all the time dependence has been incorporated into the propagator, so all we need to do is specify the initial shape of the spring to get the general solution. This solution can be restored to the $|x\rangle$ basis by applying the bra $\langle x|$ to 18 and using 10:

$$\psi(x,t) = \langle x|\psi(t)\rangle \tag{21}$$

$$= \sum_m \langle x|m\rangle\cos\frac{m\pi t}{L}\langle m|\psi(0)\rangle \tag{22}$$

$$= \sqrt{\frac{2}{L}}\sum_m \sin\frac{m\pi x}{L}\cos\frac{m\pi t}{L}\langle m|\psi(0)\rangle \tag{23}$$

We still need to get rid of the final $\langle m|$ bra in the last term, which we can do by inserting a unit operator using the $|x\rangle$ basis:

$$\psi(x,t) = \sqrt{\frac{2}{L}}\sum_m \sin\frac{m\pi x}{L}\cos\frac{m\pi t}{L}\int_0^L \left\langle m\left|x'\right.\right\rangle\left\langle x'\left|\psi(0)\right.\right\rangle dx' \tag{24}$$

$$= \frac{2}{L}\sum_m \sin\frac{m\pi x}{L}\cos\frac{m\pi t}{L}\int_0^L \sin\frac{m\pi x'}{L}\psi\left(x',0\right)dx' \tag{25}$$

The last line follows from 10 because $\left\langle m\left|x'\right.\right\rangle = \left\langle x'\left|m\right.\right\rangle^*$ and $\left\langle m\left|x'\right.\right\rangle$ is real. Thus to get the final solution, we need to do the integral in the last line, which depends on the initial shape of the string.

For example, suppose the string is held at its midpoint a distance h away from the x axis, and follows a straight line on either side of the midpoint. Then the initial state is given by

$$\psi(x,0) = \begin{cases} \frac{2xh}{L} & 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & \frac{L}{2} \le x \le L \end{cases} \tag{26}$$

We then need to do the integral

$$\int_0^L \sin\frac{m\pi x}{L}\psi(x,0)\,dx = \frac{2h}{L}\left[\int_0^{L/2} x\sin\frac{m\pi x}{L}dx + \int_{L/2}^L (L-x)\sin\frac{m\pi x}{L}dx\right] \tag{27}$$

The integrals can be done by parts although it's a bit tedious, so I used Maple to get

$$\frac{hL}{\pi^2 m^2}\left[-\left(m\pi\cos\frac{\pi m}{2} - 2\sin\frac{\pi m}{2}\right) + \left(m\pi\cos\frac{\pi m}{2} + 2\sin\frac{\pi m}{2}\right)\right] = \frac{4hL}{\pi^2 m^2}\sin\frac{\pi m}{2} \tag{28}$$

Plugging this back into 25 we get the final answer:

$$\psi(x,t) = \frac{8h}{\pi^2}\sum_m \frac{1}{m^2}\sin\frac{m\pi x}{L}\cos\frac{m\pi t}{L}\sin\frac{\pi m}{2} \tag{29}$$

Each term in the sum is a normal mode, and we can see that the amplitude drops off as $1/m^2$, so higher frequencies are less prevalent in the overall motion of the string.

Notice that if we start the string off in a pure sine wave shape, this is the only mode that is ever present. That is, if, for some fixed integer n and amplitude of initial displacement h:

$$\psi(x,0) = h\sin\frac{n\pi x}{L} \tag{30}$$

then

$$\int_0^L \sin\frac{m\pi x}{L}\psi(x,0)\,dx = \frac{hL}{2}\delta_{mn} \tag{31}$$

Thus the only mode present is $m = n$, and the string's motion is

$$\psi(x,t) = h\sin\frac{n\pi x}{L}\cos\frac{n\pi t}{L} \tag{32}$$