

## LAGRANGIANS FOR HARMONIC OSCILLATORS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.1; Exercises 2.1.1 - 2.1.2.

The Euler-Lagrange equations of motion, derived from the principle of least action are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (1)$$

where  $q_i$  and  $\dot{q}_i$  are the generalized coordinates and velocities, respectively. Here are a couple of simple examples of how these equations can be used to derive equations of motion.

**Example 1.** The harmonic oscillator. We have a mass  $m$  sliding on a frictionless horizontal surface with a spring of spring constant  $k$  connected between one end of the mass and a fixed support. The horizontal displacement of the mass from its equilibrium position is given by  $x$ , with  $x < 0$  when the mass moves to the left, compressing the spring, and  $x > 0$  when it moves to the right, stretching the spring.

For systems where the potential energy  $V(q_i)$  is independent of the velocities  $\dot{q}_i$ , the Lagrangian can be written as

$$L = T - V \quad (2)$$

where  $T$  is the kinetic energy. In the case of the mass

$$T = \frac{1}{2}m\dot{x}^2 \quad (3)$$

$$V = \frac{1}{2}kx^2 \quad (4)$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (5)$$

described earlier

The equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0 \quad (6)$$

$$m\ddot{x} = -kx \quad (7)$$

which is the familiar equation for the force on the mass equal to  $-kx$ .

**Example 2.** We can revisit the problem of two masses coupled by three springs, as described earlier. In this case, we have two coordinates  $x_1$  and  $x_2$ . The total kinetic energy is

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \quad (8)$$

The total potential energy is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}kx_2^2 \quad (9)$$

$$= k(x_1^2 + x_2^2 - x_1x_2) \quad (10)$$

The Lagrangian and equations of motion are then

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2 - x_1x_2) \quad (11)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad (12)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \quad (13)$$

This gives the same equations of motion we had earlier.

$$\ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2 \quad (14)$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2 \quad (15)$$

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