LAGRANGIANS FOR HARMONIC OSCILLATORS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.1; Exercises 2.1.1 - 2.1.2.

The Euler-Lagrange equations of motion, derived from the principle of least action are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \tag{1}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. Here are a couple of simple examples of how these equations can be used to derive equations of motion.

Example 1. The harmonic oscillator. We have a mass m sliding on a frictionless horizontal surface with a spring of spring constant k connected between one end of the mass and a fixed support. The horizontal displacement of the mass from its equilibrium position is given by x, with $x<0$ when the mass moves to the left, compressing the spring, and $x>0$ when it moves to the right, stretching the spring.

For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L=T-V \tag{2}$$

where T is the kinetic energy. In the case of the mass

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 && (3)\\ V &= \frac{1}{2}kx^2 && (4)\\ L &= \frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2 && (5) \end{aligned}$$

described earlier

The equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i} = m\ddot{x}+kx=0 \tag{6}$$

$$m\ddot{x} = -kx \tag{7}$$

which is the familiar equation for the force on the mass equal to $-kx$.

Example 2. We can revisit the problem of two masses coupled by three springs, as described earlier. In this case, we have two coordinates x_1 and x_2. The total kinetic energy is

$$T=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right) \tag{8}$$

The total potential energy is

$$V = \frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2-x_1\right)^2+\frac{1}{2}kx_2^2 \tag{9}$$

$$= k\left(x_1^2+x_2^2-x_1x_2\right) \tag{10}$$

The Lagrangian and equations of motion are then

$$L = \frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-k\left(x_1^2+x_2^2-x_1x_2\right) \tag{11}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}-\frac{\partial L}{\partial x_1} = m\ddot{x}_1+2kx_1-kx_2=0 \tag{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}-\frac{\partial L}{\partial x_2} = m\ddot{x}_2+2kx_2-kx_1=0 \tag{13}$$

This gives the same equations of motion we had earlier.

$$\ddot{x}_1 = -2\frac{k}{m}x_1+\frac{k}{m}x_2 \tag{14}$$

$$\ddot{x}_2 = \frac{k}{m}x_1-2\frac{k}{m}x_2 \tag{15}$$

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