

LAGRANGIAN FOR A SPHERICALLY SYMMETRIC POTENTIAL ENERGY FUNCTION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.1; Exercise 2.1.3.

We now consider a more general example of the Euler-Lagrange equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (1)$$

where q_i and \dot{q}_i are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities \dot{q}_i , the Lagrangian can be written as

$$L = T - V \quad (2)$$

where T is the kinetic energy.

Suppose we consider a system in three dimensions and use spherical coordinates to represent the position of a particle of mass m . We'll restrict ourselves to potential energy functions that depend only on the radial distance r from the origin, so that $V(r, \theta, \phi) = V(r)$. To write down the Lagrangian, we need an expression for the kinetic energy T .

An infinitesimal line element in spherical coordinates has a length ds given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (3)$$

The square of the velocity is then given by dividing this expression through by dt^2 , and using a dot above a symbol to indicate the derivative with respect to time t . We have

$$v^2 = \left(\frac{ds}{dt} \right)^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \quad (4)$$

The Lagrangian is then

$$L = T - V \quad (5)$$

$$= \frac{1}{2}m [\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2] - V(r) \quad (6)$$

We now get three equations of motion by applying 1 to each coordinate in turn. For r :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad (7)$$

$$\dot{r} = r\dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 - \frac{1}{m} \frac{dV}{dr} \quad (8)$$

For θ :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \quad (9)$$

$$\frac{d}{dt} (mr^2 \dot{\theta}) = mr^2 \sin \theta \cos \theta \dot{\phi}^2 \quad (10)$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r^2 \sin \theta \cos \theta \dot{\phi}^2 \quad (11)$$

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin \theta \cos \theta \dot{\phi}^2 \quad (12)$$

For ϕ :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \quad (13)$$

$$\frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = 0 \quad (14)$$

$$2r\dot{r} \sin^2 \theta \dot{\phi} + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0 \quad (15)$$

$$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2 \cot \theta \dot{\theta} \dot{\phi} \quad (16)$$

Although the only equation in which the potential energy V has a direct effect is the one for r , these three equations constitute a system of non-linear coupled differential equations so in the general case, they can be difficult to solve.

One important special case is that of a path that lies in the plane $\theta = \frac{\pi}{2}$, such as the orbit of a planet around the sun. In that case $\dot{\theta} = 0$, $\sin \theta = 1$ and $\cos \theta = 0$, so the equations simplify to

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$$\ddot{r} = r\dot{\phi}^2 - \frac{1}{m} \frac{dV}{dr} \quad (17)$$

$$\ddot{\theta} = 0 \quad (18)$$

$$\ddot{\phi} = -\frac{2}{r} \dot{r}\dot{\phi} \quad (19)$$