LAGRANGIAN FOR A SPHERICALLY SYMMETRIC POTENTIAL ENERGY FUNCTION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.1; Exercise 2.1.3.

We now consider a more general example of the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \tag{1}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L=T-V \tag{2}$$

where T is the kinetic energy.

Suppose we consider a system in three dimensions and use spherical coordinates to represent the position of a particle of mass m. We'll restrict ourselves to potential energy functions that depend only on the radial distance r from the origin, so that $V(r,\theta,\phi)=V(r)$. To write down the Lagrangian, we need an expression for the kinetic energy T.

An infinitesimal line element in spherical coordinates has a length ds given by

$$ds^2=dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2 \tag{3}$$

The square of the velocity is then given by dividing this expression through by dt^2, and using a dot above a symbol to indicate the derivative with respect to time t. We have

$$v^2=\left(\frac{ds}{dt}\right)^2=\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2 \tag{4}$$

The Lagrangian is then

$$L=T-V \tag{5}$$

$$=\frac{1}{2}m\left[\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right]-V(r) \tag{6}$$

We now get three equations of motion by applying 1 to each coordinate in turn. For r:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \tag{7}$$

$$\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 - \frac{1}{m}\frac{dV}{dr} \tag{8}$$

For θ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \tag{9}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\theta\dot{\phi}^2 \tag{10}$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r^2\sin\theta\cos\theta\dot{\phi}^2 \tag{11}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}^2 \tag{12}$$

For ϕ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \tag{13}$$

$$\frac{d}{dt}\left(mr^2\sin^2\theta\,\dot{\phi}\right) = 0 \tag{14}$$

$$2r\dot{r}\sin^2\theta\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^2\theta\ddot{\phi} = 0 \tag{15}$$

$$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\cot\theta\dot{\theta}\dot{\phi} \tag{16}$$

Although the only equation in which the potential energy V has a direct effect is the one for r, these three equations constitute a system of non-linear coupled differential equations so in the general case, they can be difficult to solve.

One important special case is that of a path that lies in the plane $\theta = \frac{\pi}{2}$, such as the orbit of a planet around the sun. In that case $\dot{\theta} = 0$, $\sin\theta = 1$ and $\cos\theta = 0$, so the equations simplify to

$$\begin{aligned} \ddot{r} &= r\dot{\phi}^2 - \frac{1}{m}\frac{dV}{dr} && (17)\\ \ddot{\theta} &= 0 && (18)\\ \ddot{\phi} &= -\frac{2}{r}\dot{r}\dot{\phi} && (19) \end{aligned}$$