

HAMILTONIANS FOR HARMONIC OSCILLATORS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.5; Exercises 2.5.2 - 2.5.3.

Here are a couple of examples of equations of motion using the Hamiltonian formalism. First, we look at the simple harmonic oscillator, in which we have a mass m sliding on a frictionless horizontal surface. The mass is connected to a spring with constant k , with the other end of the spring connected to a fixed support.

The Hamiltonian is given by

$$H(q, p) = \sum_i p_i \dot{q}_i - L(q, \dot{q}) \quad (1)$$

where the velocities \dot{q}_i are expressed in terms of the positions q_i and momenta p_i . In this case, we have, using the coordinate x as the displacement from equilibrium

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (2)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad (3)$$

$$\dot{x} = \frac{p}{m} \quad (4)$$

$$L(x, \dot{x}(x, p)) = \frac{p^2}{2m} - \frac{1}{2}kx^2 \quad (5)$$

$$H = \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{1}{2}kx^2 \right) \quad (6)$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad (7)$$

We can now apply Hamilton's canonical equations:

$$\frac{\partial H}{\partial p} = \dot{x} \quad (8)$$

$$-\frac{\partial H}{\partial x} = \dot{p} \quad (9)$$

We get

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \quad (10)$$

$$-\frac{\partial H}{\partial x} = -kx = \dot{p} \quad (11)$$

We thus get a pair of first order ODEs which can be solved in the usual way, given $x(0)$ and $p(0)$. The second order ODE that we got by using the Lagrangian method can be obtained by differentiating the first equation and plugging it into the second:

$$\ddot{x} = \frac{\dot{p}}{m} \quad (12)$$

$$= -\frac{k}{m}x \quad (13)$$

From 7 we see that, since in the absence of external force, the total energy $H = T + V = E$ is a constant,

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E = \text{constant} \quad (14)$$

This can be written as the equation of an ellipse:

$$\frac{p^2}{b^2} + \frac{x^2}{a^2} = 1 \quad (15)$$

where

$$a^2 = \frac{2E}{k} \quad (16)$$

$$b^2 = 2mE \quad (17)$$

We can use the Hamiltonian formalism to get the equations of motion of the coupled harmonic oscillator. From our Lagrangian treatment, we had

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2 - x_1x_2) \quad (18)$$

Converting to coordinates and momenta, we have

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \quad (19)$$

$$\dot{x}_i = \frac{p_i}{m} \quad (20)$$

$$H = \sum_i p_i \dot{x}_i - L(x, \dot{x}) \quad (21)$$

$$= \frac{1}{m} (p_1^2 + p_2^2) - \left[\frac{1}{2m} m (p_1^2 + p_2^2) - k (x_1^2 + x_2^2 - x_1 x_2) \right] \quad (22)$$

$$= \frac{1}{2m} (p_1^2 + p_2^2) + k (x_1^2 + x_2^2 - x_1 x_2) \quad (23)$$

Applying the canonical equations gives

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m} = \dot{x}_i \quad (24)$$

$$-\frac{\partial H}{\partial x_1} = -2kx_1 + kx_2 = \dot{p}_1 \quad (25)$$

$$-\frac{\partial H}{\partial x_2} = -2kx_2 + kx_1 = \dot{p}_2 \quad (26)$$

Again, by taking the derivative of the first line and substituting into the last two lines, we get back the previous equations of motion:

$$\ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2 \quad (27)$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2 \quad (28)$$