HAMILTONIANS FOR HARMONIC OSCILLATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.5; Exercises 2.5.2 - 2.5.3.

Here are a couple of examples of equations of motion using the Hamiltonian formalism. First, we look at the simple harmonic oscillator, in which we have a mass m sliding on a frictionless horizontal surface. The mass is connected to a spring with constant k, with the other end of the spring connected to a fixed support.

The Hamiltonian is given by

$$H(q,p)=\sum_i p_i\dot{q}_i - L(q,\dot{q}) \tag{1}$$

where the velocities $\dot{q}_i$ are expressed in terms of the positions q_i and momenta p_i. In this case, we have, using the coordinate x as the displacement from equilibrium

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2 \tag{2}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{3}$$

$$\dot{x} = \frac{p}{m} \tag{4}$$

$$L(x,\dot{x}(x,p)) = \frac{p^2}{2m}-\frac{1}{2}kx^2 \tag{5}$$

$$H = \frac{p^2}{m}-\left(\frac{p^2}{2m}-\frac{1}{2}kx^2\right) \tag{6}$$

$$= \frac{p^2}{2m}+\frac{1}{2}kx^2 \tag{7}$$

We can now apply Hamilton's canonical equations:

$$\frac{\partial H}{\partial p} = \dot{x} \tag{8}$$

$$-\frac{\partial H}{\partial x} = \dot{p} \tag{9}$$

We get

$$\begin{aligned}
\frac{\partial H}{\partial p} &= \frac{p}{m} = \dot{x} &(10)\\
-\frac{\partial H}{\partial x} &= -kx = \dot{p} &(11)
\end{aligned}$$

We thus get a pair of first order ODEs which can be solved in the usual way, given $x(0)$ and $p(0)$. The second order ODE that we got by using the Lagrangian method can be obtained by differentiating the first equation and plugging it into the second:

$$\begin{aligned}
\ddot{x} &= \frac{\dot{p}}{m} &(12)\\
&= -\frac{k}{m}x &(13)
\end{aligned}$$

From 7 we see that, since in the absence of external force, the total energy $H = T + V = E$ is a constant,

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E = \text{constant} \tag{14}$$

This can be written as the equation of an ellipse:

$$\frac{p^2}{b^2} + \frac{x^2}{a^2} = 1 \tag{15}$$

where

$$\begin{aligned}
a^2 &= \frac{2E}{k} &(16)\\
b^2 &= 2mE &(17)
\end{aligned}$$

We can use the Hamiltonian formalism to get the equations of motion of the coupled harmonic oscillator. From our Lagrangian treatment, we had

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - k\left(x_1^2 + x_2^2 - x_1x_2\right) \tag{18}$$

Converting to coordinates and momenta, we have

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \tag{19}$$

$$\dot{x}_i = \frac{p_i}{m} \tag{20}$$

$$H = \sum_i p_i \dot{x}_i - L(x,\dot{x}) \tag{21}$$

$$= \frac{1}{m}\left(p_1^2 + p_2^2\right) - \left[\frac{1}{2m}m\left(p_1^2+p_2^2\right) - k\left(x_1^2 + x_2^2 - x_1 x_2\right)\right] \tag{22}$$

$$= \frac{1}{2m}\left(p_1^2 + p_2^2\right) + k\left(x_1^2 + x_2^2 - x_1 x_2\right) \tag{23}$$

Applying the canonical equations gives

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m} = \dot{x}_i \tag{24}$$

$$-\frac{\partial H}{\partial x_1} = -2kx_1 + kx_2 = \dot{p}_1 \tag{25}$$

$$-\frac{\partial H}{\partial x_2} = -2kx_2 + kx_1 = \dot{p}_2 \tag{26}$$

Again, by taking the derivative of the first line and substituting into the last two lines, we get back the previous equations of motion:

$$\ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2 \tag{27}$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2 \tag{28}$$