HAMILTONIAN FOR THE TWO-BODY PROBLEM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.5; Exercise 2.5.4.

Here we derive the equations of motion of the two-body problem using the Hamiltonian formalism.

The Hamiltonian is given by

$$H(q,p)=\sum_i p_i\dot{q}_i - L(q,\dot{q}) \tag{1}$$

where the velocities $\dot{q}_i$ are expressed in terms of the positions q_i and momenta p_i. In this case, we start with the Lagrangian in terms of the centre of mass position $\mathbf{r}_{CM}$ and the relative position $\mathbf{r}$ of mass 2 to mass 1.

$$\begin{aligned} L &= \frac{1}{2}(m_1+m_2)\left|\dot{\mathbf{r}}_{CM}\right|^2+\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left|\dot{\mathbf{r}}\right|^2-V(\mathbf{r}) && (2)\\ &= \frac{M}{2}\left|\dot{\mathbf{r}}_{CM}\right|^2+\frac{\mu}{2}\left|\dot{\mathbf{r}}\right|^2-V(\mathbf{r}) && (3)\end{aligned}$$

where $M=m_1+m_2$ is the total mass and $\mu=\frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

There are potentially 6 velocity components and 6 coordinate components in the Lagrangian, but the 3 components of $\mathbf{r}_{CM}$ do not appear, which simplifies things a bit. To convert to a Hamiltonian, we need the momenta

$$p_i=\frac{\partial L}{\partial \dot{q}_i} \tag{4}$$

The x component of momentum of the centre of mass is

$$p_{CM,x}=\frac{\partial L}{\partial \dot{r}_{CM,x}}=M\dot{r}_{CM,x} \tag{5}$$

The other two components of the centre of mass velocity, and of the relative velocity, have a similar form, and in general we can write

$$\begin{aligned} p_{CM,i} &= M\dot{r}_{CM,i} && (6)\\ p_i &= \mu\dot{r}_i && (7)\end{aligned}$$

In vector notation, this becomes

$$\dot{\mathbf{r}}_{CM} = \frac{\mathbf{p}_{CM}}{M} \tag{8}$$
$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{\mu} \tag{9}$$
$$|\dot{\mathbf{r}}_{CM}|^2 = \frac{|\mathbf{p}_{CM}|^2}{M^2} \tag{10}$$
$$|\dot{\mathbf{r}}|^2 = \frac{|\mathbf{p}|^2}{\mu^2} \tag{11}$$

The Lagrangian thus becomes

$$L = \frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r}) \tag{12}$$

The Hamiltonian is

$$H = \mathbf{p}\cdot\dot{\mathbf{r}} + \mathbf{p}_{CM}\cdot\dot{\mathbf{r}}_{CM} - L \tag{13}$$
$$= \frac{|\mathbf{p}|^2}{\mu} + \frac{|\mathbf{p}_{CM}|^2}{M} - \left[\frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r})\right] \tag{14}$$
$$= \frac{|\mathbf{p}_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + V(\mathbf{r}) \tag{15}$$

Once we've got the Hamiltonian, we can apply Hamilton's canonical equations to get the equations of motion.

$$\frac{\partial H}{\partial p_i} = \dot{r}_i \tag{16}$$
$$-\frac{\partial H}{\partial r_i} = \dot{p}_i \tag{17}$$

Since $\mathbf{r}_{CM}$ does not appear in the Hamiltonian, we have

$$\dot{\mathbf{p}}_{CM} = 0 \tag{18}$$
$$\mathbf{p}_{CM} = \text{constant} \tag{19}$$

so the momentum of the centre of mass does not change, as expected.

For $\mathbf{r}$, we have

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{\mu} = \dot{r}_i \tag{20}$$

$$\frac{\partial H}{\partial r_i} = \frac{\partial V}{\partial r_i} = -\dot{p}_i \tag{21}$$

The first equation tells us nothing new, while the second is just Newton's law for a central force: $\dot{\mathbf{p}} = -\nabla V$.

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