CYCLIC COORDINATES AND POISSON BRACKETS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.1 - 2.7.2.

Hamilton's canonical equations are:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{1}$$

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i \tag{2}$$

If a coordinate q_i is missing in the Hamiltonian (that is, H is indepedent of q_i), then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \tag{3}$$

Thus the conjugate momentum p_i is conserved. Such a missing coordinate q_i is known as a *cyclic coordinate*. [I'm not sure of the origin of this term. Again Google doesn't provide a definitive answer.]

There is a general method for calculating the rate of change of some function $\omega(p,q)$ that depends on the momenta and coordinates, but not explicitly on the time (ω is allowed to depend implicitly on time since p and/or q can depend on time). The time derivative can then be written using the chain rule:

$$\frac{d\omega}{dt} = \sum_i \left(\frac{\partial\omega}{\partial q_i}\dot{q}_i + \frac{\partial\omega}{\partial p_i}\dot{p}_i\right) \tag{4}$$

$$= \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{5}$$

$$\equiv \{\omega, H\} \tag{6}$$

where in the second line we used Hamilton's equations 1 and 2. The last line defines the *Poisson bracket* of the function ω with the Hamiltonian H. We can see that if $\{\omega,H\} = 0$, the function ω is conserved.

Since $\{H,H\} = 0$ automatically, the total energy (represented by the Hamiltonian) is conserved, provided there is no explicit time dependence.

Such a time dependence can arise if the system is subject to some external force, for example.

From the definition 5 we can derive a few fundamental properties of Poisson brackets. We'll consider a general Poisson bracket between two arbitrary functions $\omega(p,q)$ and $\lambda(p,q)$. Then

$$\{\omega,\lambda\} = \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right) \tag{7}$$

$$= -\sum_i \left(\frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i} - \frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i}\right) \tag{8}$$

$$= -\sum_i \left(\frac{\partial\lambda}{\partial q_i}\frac{\partial\omega}{\partial p_i} - \frac{\partial\lambda}{\partial p_i}\frac{\partial\omega}{\partial q_i}\right) \tag{9}$$

$$= -\{\lambda,\omega\} \tag{10}$$

A Poisson bracket is distributive, in the sense that

$$\{\omega,\lambda+\sigma\} = \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial(\lambda+\sigma)}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial(\lambda+\sigma)}{\partial q_i}\right) \tag{11}$$

$$= \sum_i \left(\frac{\partial\omega}{\partial q_i}\left[\frac{\partial\lambda}{\partial p_i} + \frac{\partial\sigma}{\partial p_i}\right] - \frac{\partial\omega}{\partial p_i}\left[\frac{\partial\lambda}{\partial q_i} + \frac{\partial\sigma}{\partial q_i}\right]\right) \tag{12}$$

$$= \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right) + \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial q_i}\right) \tag{13}$$

$$= \{\omega,\lambda\} + \{\omega,\sigma\} \tag{14}$$

One more identity is useful, which we can derive using the product rule:

$$\{\omega,\lambda\sigma\} = \sum_i \left(\frac{\partial\omega}{\partial q_i}\frac{\partial(\lambda\sigma)}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial(\lambda\sigma)}{\partial q_i}\right) \tag{15}$$

$$= \sum_i \sigma\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right) + \sum_i \lambda\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_i} - \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial q_i}\right) \tag{16}$$

$$= \{\omega,\lambda\}\sigma + \{\omega,\sigma\}\lambda \tag{17}$$

The Poisson brackets involving the coordinates q_i and momenta p_i turn up frequently, so it's worth deriving them in detail. We have

$$\{q_i,q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right) = 0 \tag{18}$$

This follows because, in the Hamiltonian formalism, the q_is and p_is are independent variables, so $\frac{\partial q_j}{\partial p_k} = \frac{\partial p_j}{\partial q_k} = 0$ for all j and k. For the same reason, we have

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \tag{19}$$

The mixed Poisson bracket is a different story, however:

$$\{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \tag{20}$$

$$= \sum_k \delta_{ik} \delta_{jk} - 0 \tag{21}$$

$$= \delta_{ij} \tag{22}$$

Hamilton's equations 1 and 2 can be written using Poisson brackets by setting ω equal to q_i and p_i respectively in 6:

$$\dot{q}_i = \{q_i, H\} \tag{23}$$

$$\dot{p}_i = \{p_i, H\} \tag{24}$$

Example. In two dimensions, we have a Hamiltonian:

$$H = p_x^2 + p_y^2 + ax^2 + by^2 \tag{25}$$

If $a = b$, then in polar coordiantes, the only coordinate appearing in H is the radial distance from the origin $r = \sqrt{x^2 + y^2}$, which means that the polar angle θ is a cyclic coordinate. This means that the conjugate momentum p_θ must be conserved. That is,

$$\dot{p}_\theta = \{p_\theta, H\} = 0 \tag{26}$$

However, p_θ is the angular momentum ℓ_z, so this just says that angular momentum is conserved.

To see this explicitly, it's easier to convert to polar coordinates. From Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = 2p_x \tag{27}$$
$$\dot{y} = 2p_y \tag{28}$$
$$p_x^2 + p_y^2 = \frac{1}{4}\left(\dot{x}^2 + \dot{y}^2\right) \tag{29}$$
$$= \frac{v^2}{4} \tag{30}$$
$$= \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{31}$$

where in the fourth line, v is the linear velocity and in the fifth line we converted this to polar coordinates. Thus the Hamiltonian becomes, in the case where $a = b$:

$$H = \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + ar^2 \tag{32}$$

To find the conjugate momenta in polar coordinates, we can write out the Lagrangian. We use $p_x\dot{x} = \frac{\dot{x}^2}{2}$ and $p_y\dot{y} = \frac{\dot{y}^2}{2}$ and get

$$L = \sum_i p_i\dot{q}_i - H \tag{33}$$
$$= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - ar^2 \tag{34}$$
$$= \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - ar^2 \tag{35}$$

The conjugate momenta are thus

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}r^2\dot{\theta} \tag{36}$$
$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{2} \tag{37}$$

From this we can see that p_θ is indeed angular momentum as it's proportional to the product of r and the tangential velocity $v_\theta = r\dot{\theta}$. ('Real' momentum and angular momentum must, of course, also contain a factor of a mass, but from the definition of the Hamiltonian above, we see that the mass has been incorporated into the momentum parameters.)

Plugging these back into 32 we get

$$H = p_r^2 + p_\theta^2 + ar^2 \tag{38}$$

We can now calculate the Poisson brackets easily:

$$\{p_\theta, H\} = \sum_i \left(\frac{\partial p_\theta}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_\theta}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{39}$$

$$= 0 - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial H}{\partial \theta} = 0 \tag{40}$$

$$\{p_r, H\} = \sum_i \left(\frac{\partial p_r}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_r}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{41}$$

$$= 0 - \frac{\partial p_r}{\partial p_r}\frac{\partial H}{\partial r} \tag{42}$$

$$= -2ar \tag{43}$$

Thus p_θ (the angular momentum) is conserved, while $p_r < 0$, so that the object is always being pulled in towards the origin.

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