

CYCLIC COORDINATES AND POISSON BRACKETS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.1 - 2.7.2.

Hamilton's canonical equations are:

$$(1) \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$(2) \quad -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

If a coordinate q_i is missing in the Hamiltonian (that is, H is independent of q_i), then

$$(3) \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0$$

Thus the conjugate momentum p_i is conserved. Such a missing coordinate q_i is known as a *cyclic coordinate*. [I'm not sure of the origin of this term. Again Google doesn't provide a definitive answer.]

There is a general method for calculating the rate of change of some function $\omega(p, q)$ that depends on the momenta and coordinates, but not explicitly on the time (ω is allowed to depend implicitly on time since p and/or q can depend on time). The time derivative can then be written using the chain rule:

$$(4) \quad \frac{d\omega}{dt} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \dot{q}_i + \frac{\partial \omega}{\partial p_i} \dot{p}_i \right)$$

$$(5) \quad = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$(6) \quad \equiv \{\omega, H\}$$

where in the second line we used Hamilton's equations 1 and 2. The last line defines the *Poisson bracket* of the function ω with the Hamiltonian H . We can see that if $\{\omega, H\} = 0$, the function ω is conserved.

Since $\{H, H\} = 0$ automatically, the total energy (represented by the Hamiltonian) is conserved, provided there is no explicit time dependence.

Such a time dependence can arise if the system is subject to some external force, for example.

From the definition 5 we can derive a few fundamental properties of Poisson brackets. We'll consider a general Poisson bracket between two arbitrary functions $\omega(p, q)$ and $\lambda(p, q)$. Then

$$\begin{aligned}
 (7) \quad \{\omega, \lambda\} &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \\
 (8) \quad &= - \sum_i \left(\frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} \right) \\
 (9) \quad &= - \sum_i \left(\frac{\partial \lambda}{\partial q_i} \frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \omega}{\partial q_i} \right) \\
 (10) \quad &= - \{\lambda, \omega\}
 \end{aligned}$$

A Poisson bracket is distributive, in the sense that

$$\begin{aligned}
 (11) \quad \{\omega, \lambda + \sigma\} &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda + \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial (\lambda + \sigma)}{\partial q_i} \right) \\
 (12) \quad &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \left[\frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i} \right] - \frac{\partial \omega}{\partial p_i} \left[\frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i} \right] \right) \\
 (13) \quad &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \\
 (14) \quad &= \{\omega, \lambda\} + \{\omega, \sigma\}
 \end{aligned}$$

One more identity is useful, which we can derive using the product rule:

$$\begin{aligned}
 (15) \quad \{\omega, \lambda \sigma\} &= \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial (\lambda \sigma)}{\partial q_i} \right) \\
 (16) \quad &= \sum_i \sigma \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \lambda \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \\
 (17) \quad &= \{\omega, \lambda\} \sigma + \{\omega, \sigma\} \lambda
 \end{aligned}$$

The Poisson brackets involving the coordinates q_i and momenta p_i turn up frequently, so it's worth deriving them in detail. We have

$$(18) \quad \{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0$$

This follows because, in the Hamiltonian formalism, the q_i s and p_i s are independent variables, so $\frac{\partial q_j}{\partial p_k} = \frac{\partial p_j}{\partial q_k} = 0$ for all j and k . For the same reason, we have

$$(19) \quad \{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0$$

The mixed Poisson bracket is a different story, however:

$$(20) \quad \{q_i, p_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right)$$

$$(21) \quad = \sum_k \delta_{ik} \delta_{jk} - 0$$

$$(22) \quad = \delta_{ij}$$

Hamilton's equations 1 and 2 can be written using Poisson brackets by setting ω equal to q_i and p_i respectively in 6:

$$(23) \quad \dot{q}_i = \{q_i, H\}$$

$$(24) \quad \dot{p}_i = \{p_i, H\}$$

Example. In two dimensions, we have a Hamiltonian:

$$(25) \quad H = p_x^2 + p_y^2 + ax^2 + by^2$$

If $a = b$, then in polar coordinates, the only coordinate appearing in H is the radial distance from the origin $r = \sqrt{x^2 + y^2}$, which means that the polar angle θ is a cyclic coordinate. This means that the conjugate momentum p_θ must be conserved. That is,

$$(26) \quad \dot{p}_\theta = \{p_\theta, H\} = 0$$

However, p_θ is the angular momentum ℓ_z , so this just says that angular momentum is conserved.

To see this explicitly, it's easier to convert to polar coordinates. From Hamilton's equations

$$(27) \quad \dot{x} = \frac{\partial H}{\partial p_x} = 2p_x$$

$$(28) \quad \dot{y} = 2p_y$$

$$(29) \quad p_x^2 + p_y^2 = \frac{1}{4}(\dot{x}^2 + \dot{y}^2)$$

$$(30) \quad = \frac{v^2}{4}$$

$$(31) \quad = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2)$$

where in the fourth line, v is the linear velocity and in the fifth line we converted this to polar coordinates. Thus the Hamiltonian becomes, in the case where $a = b$:

$$(32) \quad H = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) + ar^2$$

To find the conjugate momenta in polar coordinates, we can write out the Lagrangian. We use $p_x\dot{x} = \frac{\dot{x}^2}{2}$ and $p_y\dot{y} = \frac{\dot{y}^2}{2}$ and get

$$(33) \quad L = \sum_i p_i\dot{q}_i - H$$

$$(34) \quad = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2$$

$$(35) \quad = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2$$

The conjugate momenta are thus

$$(36) \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}r^2\dot{\theta}$$

$$(37) \quad p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{2}$$

From this we can see that p_θ is indeed angular momentum as it's proportional to the product of r and the tangential velocity $v_\theta = r\dot{\theta}$. ('Real' momentum and angular momentum must, of course, also contain a factor of a mass, but from the definition of the Hamiltonian above, we see that the mass has been incorporated into the momentum parameters.)

Plugging these back into 32 we get

$$(38) \quad H = p_r^2 + p_\theta^2 + ar^2$$

We can now calculate the Poisson brackets easily:

$$(39) \quad \{p_\theta, H\} = \sum_i \left(\frac{\partial p_\theta}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_\theta}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$(40) \quad = 0 - \frac{\partial p_\theta}{\partial p_\theta} \frac{\partial H}{\partial \theta} = 0$$

$$(41) \quad \{p_r, H\} = \sum_i \left(\frac{\partial p_r}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_r}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$(42) \quad = 0 - \frac{\partial p_r}{\partial p_r} \frac{\partial H}{\partial r}$$

$$(43) \quad = -2ar$$

Thus p_θ (the angular momentum) is conserved, while $p_r < 0$, so that the object is always being pulled in towards the origin.

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