

## CYCLIC COORDINATES AND POISSON BRACKETS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.1 - 2.7.2.

Hamilton's canonical equations are:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad (1)$$

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i \quad (2)$$

If a coordinate  $q_i$  is missing in the Hamiltonian (that is,  $H$  is independent of  $q_i$ ), then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \quad (3)$$

Thus the conjugate momentum  $p_i$  is conserved. Such a missing coordinate  $q_i$  is known as a *cyclic coordinate*. [I'm not sure of the origin of this term. Again Google doesn't provide a definitive answer.]

There is a general method for calculating the rate of change of some function  $\omega(p, q)$  that depends on the momenta and coordinates, but not explicitly on the time ( $\omega$  is allowed to depend implicitly on time since  $p$  and/or  $q$  can depend on time). The time derivative can then be written using the chain rule:

$$\frac{d\omega}{dt} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \dot{q}_i + \frac{\partial \omega}{\partial p_i} \dot{p}_i \right) \quad (4)$$

$$= \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (5)$$

$$\equiv \{\omega, H\} \quad (6)$$

where in the second line we used Hamilton's equations 1 and 2. The last line defines the *Poisson bracket* of the function  $\omega$  with the Hamiltonian  $H$ . We can see that if  $\{\omega, H\} = 0$ , the function  $\omega$  is conserved.

Since  $\{H, H\} = 0$  automatically, the total energy (represented by the Hamiltonian) is conserved, provided there is no explicit time dependence.

Such a time dependence can arise if the system is subject to some external force, for example.

From the definition 5 we can derive a few fundamental properties of Poisson brackets. We'll consider a general Poisson bracket between two arbitrary functions  $\omega(p, q)$  and  $\lambda(p, q)$ . Then

$$\{\omega, \lambda\} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \quad (7)$$

$$= - \sum_i \left( \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} \right) \quad (8)$$

$$= - \sum_i \left( \frac{\partial \lambda}{\partial q_i} \frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \omega}{\partial q_i} \right) \quad (9)$$

$$= - \{\lambda, \omega\} \quad (10)$$

A Poisson bracket is distributive, in the sense that

$$\{\omega, \lambda + \sigma\} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda + \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial (\lambda + \sigma)}{\partial q_i} \right) \quad (11)$$

$$= \sum_i \left( \frac{\partial \omega}{\partial q_i} \left[ \frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i} \right] - \frac{\partial \omega}{\partial p_i} \left[ \frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i} \right] \right) \quad (12)$$

$$= \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \quad (13)$$

$$= \{\omega, \lambda\} + \{\omega, \sigma\} \quad (14)$$

One more identity is useful, which we can derive using the product rule:

$$\{\omega, \lambda \sigma\} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial (\lambda \sigma)}{\partial q_i} \right) \quad (15)$$

$$= \sum_i \sigma \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \lambda \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) \quad (16)$$

$$= \{\omega, \lambda\} \sigma + \{\omega, \sigma\} \lambda \quad (17)$$

The Poisson brackets involving the coordinates  $q_i$  and momenta  $p_i$  turn up frequently, so it's worth deriving them in detail. We have

$$\{q_i, q_j\} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0 \quad (18)$$

This follows because, in the Hamiltonian formalism, the  $q_i$ s and  $p_i$ s are independent variables, so  $\frac{\partial q_j}{\partial p_k} = \frac{\partial p_j}{\partial q_k} = 0$  for all  $j$  and  $k$ . For the same reason, we have

$$\{p_i, p_j\} = \sum_k \left( \frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \quad (19)$$

The mixed Poisson bracket is a different story, however:

$$\{q_i, p_j\} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \quad (20)$$

$$= \sum_k \delta_{ik} \delta_{jk} - 0 \quad (21)$$

$$= \delta_{ij} \quad (22)$$

Hamilton's equations 1 and 2 can be written using Poisson brackets by setting  $\omega$  equal to  $q_i$  and  $p_i$  respectively in 6:

$$\dot{q}_i = \{q_i, H\} \quad (23)$$

$$\dot{p}_i = \{p_i, H\} \quad (24)$$

**Example.** In two dimensions, we have a Hamiltonian:

$$H = p_x^2 + p_y^2 + ax^2 + by^2 \quad (25)$$

If  $a = b$ , then in polar coordinates, the only coordinate appearing in  $H$  is the radial distance from the origin  $r = \sqrt{x^2 + y^2}$ , which means that the polar angle  $\theta$  is a cyclic coordinate. This means that the conjugate momentum  $p_\theta$  must be conserved. That is,

$$\dot{p}_\theta = \{p_\theta, H\} = 0 \quad (26)$$

However,  $p_\theta$  is the angular momentum  $\ell_z$ , so this just says that angular momentum is conserved.

To see this explicitly, it's easier to convert to polar coordinates. From Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = 2p_x \quad (27)$$

$$\dot{y} = 2p_y \quad (28)$$

$$p_x^2 + p_y^2 = \frac{1}{4}(\dot{x}^2 + \dot{y}^2) \quad (29)$$

$$= \frac{v^2}{4} \quad (30)$$

$$= \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) \quad (31)$$

where in the fourth line,  $v$  is the linear velocity and in the fifth line we converted this to polar coordinates. Thus the Hamiltonian becomes, in the case where  $a = b$ :

$$H = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) + ar^2 \quad (32)$$

To find the conjugate momenta in polar coordinates, we can write out the Lagrangian. We use  $p_x\dot{x} = \frac{\dot{x}^2}{2}$  and  $p_y\dot{y} = \frac{\dot{y}^2}{2}$  and get

$$L = \sum_i p_i\dot{q}_i - H \quad (33)$$

$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (34)$$

$$= \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (35)$$

The conjugate momenta are thus

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}r^2\dot{\theta} \quad (36)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{2} \quad (37)$$

From this we can see that  $p_\theta$  is indeed angular momentum as it's proportional to the product of  $r$  and the tangential velocity  $v_\theta = r\dot{\theta}$ . ('Real' momentum and angular momentum must, of course, also contain a factor of a mass, but from the definition of the Hamiltonian above, we see that the mass has been incorporated into the momentum parameters.)

Plugging these back into 32 we get

$$H = p_r^2 + p_\theta^2 + ar^2 \quad (38)$$

We can now calculate the Poisson brackets easily:

$$\{p_\theta, H\} = \sum_i \left( \frac{\partial p_\theta}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_\theta}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (39)$$

$$= 0 - \frac{\partial p_\theta}{\partial p_\theta} \frac{\partial H}{\partial \theta} = 0 \quad (40)$$

$$\{p_r, H\} = \sum_i \left( \frac{\partial p_r}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_r}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (41)$$

$$= 0 - \frac{\partial p_r}{\partial p_r} \frac{\partial H}{\partial r} \quad (42)$$

$$= -2ar \quad (43)$$

Thus  $p_\theta$  (the angular momentum) is conserved, while  $p_r < 0$ , so that the object is always being pulled in towards the origin.

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