

CONDITIONS FOR A TRANSFORMATION TO BE CANONICAL

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercise 2.7.3.

We've seen that the Euler-Lagrange equations are invariant under canonical transformations, but in the Hamiltonian formalism where the system moves in a $2n$ -dimensional phase space with n coordinates q and n momenta p , more general transformations are possible:

$$\bar{q}_i = \bar{q}_i(q, p) \quad (1)$$

$$\bar{p}_i = \bar{p}_i(q, p) \quad (2)$$

In order for such a transformation to be canonical, we require that the new variables \bar{q} and \bar{p} satisfy Hamilton's equations, that is

$$\frac{\partial H}{\partial \bar{p}_i} = \dot{\bar{q}}_i \quad (3)$$

$$-\frac{\partial H}{\partial \bar{q}_i} = \dot{\bar{p}}_i \quad (4)$$

In principle, then, we could check the Hamiltonian in the new coordinates to see if these equations are valid, but it would seem that whether or not a set of coordinates and momenta is canonical should be determinable from the variables themselves, and not depend on the specific Hamiltonian. Here we derive a set of conditions on the \bar{q} and \bar{p} that determine whether or not the transformation is canonical.

The time derivative of any function ω can be written as a Poisson bracket:

$$\dot{\omega} = \{\omega, H\} \quad (5)$$

For the transformed velocities, we have

$$\dot{\bar{q}}_j = \{\bar{q}_j, H\} \quad (6)$$

$$= \sum_i \left(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (7)$$

Here, H is written as a function $H(q, p)$ of the original variables. If we write it as a function of the transformed variables, we can find the two derivatives of H in 7 by using the chain rule:

$$\frac{\partial H(\bar{q}, \bar{p})}{\partial p_i} = \sum_k \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial p_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial p_i} \right) \quad (8)$$

$$\frac{\partial H(\bar{q}, \bar{p})}{\partial q_i} = \sum_k \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial q_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial q_i} \right) \quad (9)$$

Inserting these into 7 we get

$$\dot{\bar{q}}_j = \sum_i \sum_k \left[\frac{\partial \bar{q}_j}{\partial q_i} \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial p_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial p_i} \right) - \frac{\partial \bar{q}_j}{\partial p_i} \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial q_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial q_i} \right) \right] \quad (10)$$

$$= \sum_k \frac{\partial H}{\partial \bar{q}_k} \sum_i \left(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \bar{q}_k}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \bar{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \bar{p}_k} \sum_i \left(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \bar{p}_k}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \bar{p}_k}{\partial q_i} \right) \quad (11)$$

$$= \sum_k \frac{\partial H}{\partial \bar{q}_k} \{ \bar{q}_j, \bar{q}_k \} + \sum_k \frac{\partial H}{\partial \bar{p}_k} \{ \bar{q}_j, \bar{p}_k \} \quad (12)$$

In order for this result to satisfy 3, we must have

$$\{ \bar{q}_j, \bar{q}_k \} = 0 \quad (13)$$

$$\{ \bar{q}_j, \bar{p}_k \} = \delta_{jk} \quad (14)$$

We can repeat the calculation for $\dot{\bar{p}}_j$:

$$\dot{\bar{p}}_j = \{ \bar{p}_j, H \} \quad (15)$$

$$= \sum_i \left(\frac{\partial \bar{p}_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \bar{p}_j}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (16)$$

$$= \sum_i \sum_k \left[\frac{\partial \bar{p}_j}{\partial q_i} \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial p_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial p_i} \right) - \frac{\partial \bar{p}_j}{\partial p_i} \left(\frac{\partial H}{\partial \bar{q}_k} \frac{\partial \bar{q}_k}{\partial q_i} + \frac{\partial H}{\partial \bar{p}_k} \frac{\partial \bar{p}_k}{\partial q_i} \right) \right] \quad (17)$$

$$= \sum_k \frac{\partial H}{\partial \bar{q}_k} \sum_i \left(\frac{\partial \bar{p}_j}{\partial q_i} \frac{\partial \bar{q}_k}{\partial p_i} - \frac{\partial \bar{p}_j}{\partial p_i} \frac{\partial \bar{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \bar{p}_k} \sum_i \left(\frac{\partial \bar{p}_j}{\partial q_i} \frac{\partial \bar{p}_k}{\partial p_i} - \frac{\partial \bar{p}_j}{\partial p_i} \frac{\partial \bar{p}_k}{\partial q_i} \right) \quad (18)$$

$$= \sum_k \frac{\partial H}{\partial \bar{q}_k} \{ \bar{p}_j, \bar{q}_k \} + \sum_k \frac{\partial H}{\partial \bar{p}_k} \{ \bar{p}_j, \bar{p}_k \} \quad (19)$$

Requiring this to satisfy 4, we have

$$\{\bar{p}_j, \bar{p}_k\} = 0 \tag{20}$$

$$\{\bar{p}_j, \bar{q}_k\} = -\delta_{jk} \tag{21}$$

The last equation is equivalent to

$$\{\bar{q}_j, \bar{p}_k\} = \delta_{jk} \tag{22}$$

which agrees with 14. Thus in order for the transformation to be canonical, the conditions are

$$\{\bar{q}_j, \bar{q}_k\} = \{\bar{p}_j, \bar{p}_k\} = 0 \tag{23}$$

$$\{\bar{q}_j, \bar{p}_k\} = \delta_{jk} \tag{24}$$

Note that these Poisson brackets require calculating the derivatives of the new variables \bar{q} and \bar{p} with respect to the original ones q and p , but they *don't* involve any particular Hamiltonian. Thus it's possible to determine whether or not a transformation is canonical entirely from the transformation equations 1 and 2.

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