

CANONICAL TRANSFORMATIONS IN 2-D: ROTATIONS AND POLAR COORDINATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.4 - 2.7.5.

Here are a couple of examples of canonical variable transformations.

Example 1. We rotate the 2-d rectangular coordinates through an angle θ , giving the transformations

$$\begin{aligned}(1) \quad & \bar{x} = x \cos \theta - y \sin \theta \\(2) \quad & \bar{y} = x \sin \theta + y \cos \theta \\(3) \quad & \bar{p}_x = p_x \cos \theta - p_y \sin \theta \\(4) \quad & \bar{p}_y = p_x \sin \theta + p_y \cos \theta\end{aligned}$$

To show this is a canonical transformation, we must evaluate the Poisson brackets. Here, $q_1 = x$ and $q_2 = y$. Remember that θ is a constant in these derivatives.

$$\begin{aligned}(5) \quad & \{\bar{x}, \bar{y}\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{y}}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{y}}{\partial q_i} \right) \\(6) \quad & = 0\end{aligned}$$

since neither coordinate depends on any momentum. Similarly $\{\bar{p}_x, \bar{p}_y\} = 0$ since this Poisson bracket contains derivatives of \bar{p}_i with respect to q_i and these are all zero. The remaining Poisson brackets are of the form $\{\bar{q}_i, \bar{p}_j\}$. There are four of these, but we'll work out only a couple. The other two have similar forms.

$$(7) \quad \{\bar{x}, \bar{p}_x\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_x}{\partial q_i} \right)$$

$$(8) \quad = \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}_x}{\partial p_x} + \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{p}_x}{\partial p_y}$$

$$(9) \quad = \cos^2 \theta + \sin^2 \theta$$

$$(10) \quad = 1$$

$$(11) \quad \{\bar{x}, \bar{p}_y\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_y}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_y}{\partial q_i} \right)$$

$$(12) \quad = \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}_y}{\partial p_x} + \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{p}_y}{\partial p_y}$$

$$(13) \quad = \sin \theta \cos \theta - \sin \theta \cos \theta$$

$$(14) \quad = 0$$

Similarly

$$(15) \quad \{\bar{y}, \bar{p}_x\} = 0$$

$$(16) \quad \{\bar{y}, \bar{p}_y\} = 1$$

Example 2. The transformation from 2-d rectangular to polar coordinates is given by

$$(17) \quad \rho = \sqrt{x^2 + y^2}$$

$$(18) \quad \phi = \arctan \frac{y}{x}$$

$$(19) \quad p_\rho = \frac{x p_x + y p_y}{\sqrt{x^2 + y^2}}$$

$$(20) \quad p_\phi = x p_y - y p_x$$

For the Poisson brackets we have

$$(21) \quad \{\rho, \phi\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial \phi}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial \phi}{\partial q_i} \right)$$

$$(22) \quad = 0$$

because, again, the coordinates don't depend on the momenta.

In this case, however, the new momenta do depend on the old coordinates, so we need to actually do some calculation.

(23)

$$\{p_\rho, p_\phi\} = \sum_i \left(\frac{\partial p_\rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial p_\rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right)$$

$$= \left(-\frac{x(xp_x + yp_y)}{(x^2 + y^2)^{3/2}} + \frac{p_x}{\sqrt{x^2 + y^2}} \right) (-y) - \frac{x}{\sqrt{x^2 + y^2}} p_y +$$

$$(24) \quad \left(-\frac{y(xp_x + yp_y)}{(x^2 + y^2)^{3/2}} + \frac{p_y}{\sqrt{x^2 + y^2}} \right) x - \frac{y}{\sqrt{x^2 + y^2}} (-p_x)$$

$$(25) \quad = -\frac{y^2(y p_x - x p_y)}{(x^2 + y^2)^{3/2}} - \frac{x}{\sqrt{x^2 + y^2}} p_y - \frac{x^2(y p_x - x p_y)}{(x^2 + y^2)^{3/2}} + \frac{y}{\sqrt{x^2 + y^2}} p_x$$

$$(26) \quad = -\frac{y^3 p_x + x^3 p_y}{(x^2 + y^2)^{3/2}} + \frac{y^3 p_x + x^3 p_y}{(x^2 + y^2)^{3/2}}$$

$$(27) \quad = 0$$

Finally, we need to work out the mixed brackets.

$$(28) \quad \{\rho, p_\rho\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial p_\rho}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial p_\rho}{\partial q_i} \right)$$

$$(29) \quad = \frac{x^2}{x^2 + y^2} - 0 + \frac{y^2}{x^2 + y^2} - 0$$

$$(30) \quad = 1$$

$$(31) \quad \{\rho, p_\phi\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right)$$

$$(32) \quad = -\frac{xy}{x^2 + y^2} - 0 + \frac{xy}{x^2 + y^2} - 0$$

$$(33) \quad = 0$$

$$(34) \quad \{\phi, p_\rho\} = \sum_i \left(\frac{\partial \phi}{\partial q_i} \frac{\partial p_\rho}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial p_\rho}{\partial q_i} \right)$$

$$(35) \quad = -\frac{y}{x \left(1 + \frac{y^2}{x^2}\right) \sqrt{x^2 + y^2}} - 0 + \frac{y}{x \left(1 + \frac{y^2}{x^2}\right) \sqrt{x^2 + y^2}} - 0$$

$$(36) \quad = 0$$

$$(37) \quad \{\phi, p_\phi\} = \sum_i \left(\frac{\partial \phi}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right)$$

$$(38) \quad = \frac{y^2}{x^2 \left(1 + \frac{y^2}{x^2}\right)} - 0 + \frac{1}{1 + \frac{y^2}{x^2}} - 0$$

$$(39) \quad = \frac{y^2}{x^2 + y^2} + \frac{x^2}{x^2 + y^2}$$

$$(40) \quad = 1$$

Thus all the Poisson brackets are correct, so the transformation is canonical.

PINGBACKS

Pingback: Infinitesimal rotations in canonical and noncanonical transformations