

CANONICAL TRANSFORMATIONS IN 2-D: ROTATIONS AND POLAR COORDINATES

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.4 - 2.7.5.

Here are a couple of examples of canonical variable transformations.

Example 1. We rotate the 2-d rectangular coordinates through an angle θ , giving the transformations

$$\bar{x} = x \cos \theta - y \sin \theta \quad (1)$$

$$\bar{y} = x \sin \theta + y \cos \theta \quad (2)$$

$$\bar{p}_x = p_x \cos \theta - p_y \sin \theta \quad (3)$$

$$\bar{p}_y = p_x \sin \theta + p_y \cos \theta \quad (4)$$

To show this is a canonical transformation, we must evaluate the Poisson brackets. Here, $q_1 = x$ and $q_2 = y$. Remember that θ is a constant in these derivatives.

$$\{\bar{x}, \bar{y}\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{y}}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{y}}{\partial q_i} \right) \quad (5)$$

$$= 0 \quad (6)$$

since neither coordinate depends on any momentum. Similarly $\{\bar{p}_x, \bar{p}_y\} = 0$ since this Poisson bracket contains derivatives of \bar{p}_i with respect to q_i and these are all zero. The remaining Poisson brackets are of the form $\{\bar{q}_i, \bar{p}_j\}$. There are four of these, but we'll work out only a couple. The other two have similar forms.

$$\{\bar{x}, \bar{p}_x\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_x}{\partial q_i} \right) \quad (7)$$

$$= \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}_x}{\partial p_x} + \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{p}_x}{\partial p_y} \quad (8)$$

$$= \cos^2 \theta + \sin^2 \theta \quad (9)$$

$$= 1 \quad (10)$$

$$\{\bar{x}, \bar{p}_y\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_y}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_y}{\partial q_i} \right) \quad (11)$$

$$= \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}_y}{\partial p_x} + \frac{\partial \bar{x}}{\partial y} \frac{\partial \bar{p}_y}{\partial p_y} \quad (12)$$

$$= \sin \theta \cos \theta - \sin \theta \cos \theta \quad (13)$$

$$= 0 \quad (14)$$

Similarly

$$\{\bar{y}, \bar{p}_x\} = 0 \quad (15)$$

$$\{\bar{y}, \bar{p}_y\} = 1 \quad (16)$$

Example 2. The transformation from 2-d rectangular to polar coordinates is given by

$$\rho = \sqrt{x^2 + y^2} \quad (17)$$

$$\phi = \arctan \frac{y}{x} \quad (18)$$

$$p_\rho = \frac{xp_x + yp_y}{\sqrt{x^2 + y^2}} \quad (19)$$

$$p_\phi = xp_y - yp_x \quad (20)$$

For the Poisson brackets we have

$$\{\rho, \phi\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial \phi}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial \phi}{\partial q_i} \right) \quad (21)$$

$$= 0 \quad (22)$$

because, again, the coordinates don't depend on the momenta.

In this case, however, the new momenta do depend on the old coordinates, so we need to actually do some calculation.

$$\{p_\rho, p_\phi\} = \sum_i \left(\frac{\partial p_\rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial p_\rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right) \quad (23)$$

$$= \left(-\frac{x(xp_x + yp_y)}{(x^2 + y^2)^{3/2}} + \frac{p_x}{\sqrt{x^2 + y^2}} \right) (-y) - \frac{x}{\sqrt{x^2 + y^2}} p_y +$$

$$\left(-\frac{y(xp_x + yp_y)}{(x^2 + y^2)^{3/2}} + \frac{p_y}{\sqrt{x^2 + y^2}} \right) x - \frac{y}{\sqrt{x^2 + y^2}} (-p_x) \quad (24)$$

$$= -\frac{y^2(y p_x - x p_y)}{(x^2 + y^2)^{3/2}} - \frac{x}{\sqrt{x^2 + y^2}} p_y - \frac{x^2(y p_x - x p_y)}{(x^2 + y^2)^{3/2}} + \frac{y}{\sqrt{x^2 + y^2}} p_x \quad (25)$$

$$= -\frac{y^3 p_x + x^3 p_y}{(x^2 + y^2)^{3/2}} + \frac{y^3 p_x + x^3 p_y}{(x^2 + y^2)^{3/2}} \quad (26)$$

$$= 0 \quad (27)$$

Finally, we need to work out the mixed brackets.

$$\{\rho, p_\rho\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial p_\rho}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial p_\rho}{\partial q_i} \right) \quad (28)$$

$$= \frac{x^2}{x^2 + y^2} - 0 + \frac{y^2}{x^2 + y^2} - 0 \quad (29)$$

$$= 1 \quad (30)$$

$$\{\rho, p_\phi\} = \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right) \quad (31)$$

$$= -\frac{xy}{\sqrt{x^2 + y^2}} - 0 + \frac{xy}{\sqrt{x^2 + y^2}} - 0 \quad (32)$$

$$= 0 \quad (33)$$

$$\{\phi, p_\rho\} = \sum_i \left(\frac{\partial \phi}{\partial q_i} \frac{\partial p_\rho}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial p_\rho}{\partial q_i} \right) \quad (34)$$

$$= -\frac{y}{x \left(1 + \frac{y^2}{x^2}\right) \sqrt{x^2 + y^2}} - 0 + \frac{y}{x \left(1 + \frac{y^2}{x^2}\right) \sqrt{x^2 + y^2}} - 0 \quad (35)$$

$$= 0 \quad (36)$$

$$\{\phi, p_\phi\} = \sum_i \left(\frac{\partial \phi}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial \phi}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right) \quad (37)$$

$$= \frac{y^2}{x^2 \left(1 + \frac{y^2}{x^2}\right)} - 0 + \frac{1}{1 + \frac{y^2}{x^2}} - 0 \quad (38)$$

$$= \frac{y^2}{x^2 + y^2} + \frac{x^2}{x^2 + y^2} \quad (39)$$

$$= 1 \quad (40)$$

Thus all the Poisson brackets are correct, so the transformation is canonical.

PINGBACKS

Pingback: Infinitesimal rotations in canonical and noncanonical transformations