CANONICAL TRANSFORMATIONS IN 2-D: ROTATIONS AND POLAR COORDINATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.4 - 2.7.5.

Here are a couple of examples of canonical variable transformations.

Example 1. We rotate the 2-d rectangular coordinates through an angle θ, giving the transformations

$$\bar{x} = x\cos\theta - y\sin\theta \tag{0.1}$$

$$\bar{y} = x\sin\theta + y\cos\theta \tag{0.2}$$

$$\overline{p}_x = p_x\cos\theta - p_y\sin\theta \tag{0.3}$$

$$\overline{p}_y = p_x\sin\theta + p_y\cos\theta \tag{0.4}$$

To show this is a canonical transformation, we must evaluate the Poisson brackets. Here, $q_1 = x$ and $q_2 = y$. Remember that θ is a constant in these derivatives.

$$\{\bar{x},\bar{y}\} = \sum_i \left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{y}}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{y}}{\partial q_i}\right) \tag{0.5}$$

$$= 0 \tag{0.6}$$

since neither coordinate depends on any momentum. Similarly $\left\{\overline{p}_x,\overline{p}_y\right\} = 0$ since this Poisson bracket contains derivatives of $\overline{p}_i$ with respect to q_i and these are all zero. The remaining Poisson bracket are of the form $\left\{\overline{q}_i,\overline{p}_j\right\}$. There are four of these, but we'll work out only a couple. The other two have similar forms.

$$\{\bar{x},\bar{p}_x\} = \sum_i \left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{p}_x}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{p}_x}{\partial q_i}\right) \tag{0.7}$$

$$= \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} \tag{0.8}$$

$$= \cos^2\theta + \sin^2\theta \tag{0.9}$$

$$= 1 \tag{0.10}$$

$$\{\bar{x},\bar{p}_y\} = \sum_i \left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{p}_y}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{p}_y}{\partial q_i}\right) \tag{0.11}$$

$$= \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x} + \frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} \tag{0.12}$$

$$= \sin\theta\cos\theta - \sin\theta\cos\theta \tag{0.13}$$

$$= 0 \tag{0.14}$$

Similarly

$$\{\bar{y},\bar{p}_x\} = 0 \tag{0.15}$$

$$\{\bar{y},\bar{p}_y\} = 1 \tag{0.16}$$

Example 2. The transformation from 2-d rectangular to polar coordinates is given by

$$\rho = \sqrt{x^2+y^2} \tag{0.17}$$

$$\phi = \arctan\frac{y}{x} \tag{0.18}$$

$$p_\rho = \frac{xp_x + yp_y}{\sqrt{x^2+y^2}} \tag{0.19}$$

$$p_\phi = xp_y - yp_x \tag{0.20}$$

For the Poisson brackets we have

$$\{\rho,\phi\} = \sum_i \left(\frac{\partial\rho}{\partial q_i}\frac{\partial\phi}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial\phi}{\partial q_i}\right) \tag{0.21}$$

$$= 0 \tag{0.22}$$

because, again, the coordinates don't depend on the momenta.

In this case, however, the new momenta do depend on the old coordinates, so we need to actually do some calculation.

(0.23)

$$\{p_\rho, p_\phi\} = \sum_i \left(\frac{\partial p_\rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial p_\rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right)$$

$$= \left(-\frac{x\,(xp_x + yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_x}{\sqrt{x^2+y^2}} \right)(-y) - \frac{x}{\sqrt{x^2+y^2}} p_y +$$

(0.24)

$$\left(-\frac{y\,(xp_x + yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_y}{\sqrt{x^2+y^2}} \right) x - \frac{y}{\sqrt{x^2+y^2}} (-p_x)$$

(0.25)

$$= -\frac{y^2\,(yp_x - xp_y)}{(x^2+y^2)^{3/2}} - \frac{x}{\sqrt{x^2+y^2}} p_y - \frac{x^2\,(yp_x - xp_y)}{(x^2+y^2)^{3/2}} + \frac{y}{\sqrt{x^2+y^2}} p_x$$

(0.26)

$$= -\frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}} + \frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}}$$

(0.27)

$$= 0$$

Finally, we need to work out the mixed brackets.

$$\{\rho, p_\rho\} = \sum_i \left(\frac{\partial\rho}{\partial q_i}\frac{\partial p_\rho}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial p_\rho}{\partial q_i} \right) \tag{0.28}$$

$$= \frac{x^2}{x^2+y^2} - 0 + \frac{y^2}{x^2+y^2} - 0 \tag{0.29}$$

$$= 1 \tag{0.30}$$

$$\{\rho, p_\phi\} = \sum_i \left(\frac{\partial\rho}{\partial q_i}\frac{\partial p_\phi}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial p_\phi}{\partial q_i} \right) \tag{0.31}$$

$$= -\frac{xy}{x^2+y^2} - 0 + \frac{xy}{x^2+y^2} - 0 \tag{0.32}$$

$$= 0 \tag{0.33}$$

$$\{\phi, p_\rho\} = \sum_i \left(\frac{\partial\phi}{\partial q_i}\frac{\partial p_\rho}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\frac{\partial p_\rho}{\partial q_i} \right) \tag{0.34}$$

$$= -\frac{y}{x\left(1+\frac{y^2}{x^2}\right)\sqrt{x^2+y^2}} - 0 + \frac{y}{x\left(1+\frac{y^2}{x^2}\right)\sqrt{x^2+y^2}} - 0 \tag{0.35}$$

$$= 0 \tag{0.36}$$

$$\{\phi, p_\phi\} = \sum_i \left(\frac{\partial\phi}{\partial q_i}\frac{\partial p_\phi}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\frac{\partial p_\phi}{\partial q_i} \right) \tag{0.37}$$

$$= \frac{y^2}{x^2\left(1+\frac{y^2}{x^2}\right)} - 0 + \frac{1}{1+\frac{y^2}{x^2}} - 0 \tag{0.38}$$

$$= \frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2} \tag{0.39}$$

$$= 1 \tag{0.40}$$

Thus all the Poisson brackets are correct, so the transformation is canonical.

PINGBACKS

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