CANONICAL TRANSFORMATIONS IN 2-D: ROTATIONS AND POLAR COORDINATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.4 - 2.7.5.

Here are a couple of examples of canonical variable transformations.

Example 1. We rotate the 2-d rectangular coordinates through an angle θ, giving the transformations

$$\bar{x} = x\cos\theta - y\sin\theta \tag{1}$$

$$\bar{y} = x\sin\theta + y\cos\theta \tag{2}$$

$$\overline{p}_x = p_x\cos\theta - p_y\sin\theta \tag{3}$$

$$\overline{p}_y = p_x\sin\theta + p_y\cos\theta \tag{4}$$

To show this is a canonical transformation, we must evaluate the Poisson brackets. Here, $q_1 = x$ and $q_2 = y$. Remember that θ is a constant in these derivatives.

$$\{\bar{x},\bar{y}\} = \sum_i\left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{y}}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{y}}{\partial q_i}\right) \tag{5}$$

$$= 0 \tag{6}$$

since neither coordinate depends on any momentum. Similarly $\left\{\overline{p}_x,\overline{p}_y\right\} = 0$ since this Poisson bracket contains derivatives of $\overline{p}_i$ with respect to q_i and these are all zero. The remaining Poisson bracket are of the form $\left\{\overline{q}_i,\overline{p}_j\right\}$. There are four of these, but we'll work out only a couple. The other two have similar forms.

$$\{\bar{x},\bar{p}_x\} = \sum_i\left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{p}_x}{\partial p_i}-\frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{p}_x}{\partial q_i}\right) \tag{7}$$

$$= \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_x}{\partial p_x}+\frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_x}{\partial p_y} \tag{8}$$

$$= \cos^2\theta+\sin^2\theta \tag{9}$$

$$= 1 \tag{10}$$

$$\left\{\bar{x},\bar{p}_y\right\} = \sum_i\left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{p}_y}{\partial p_i}-\frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{p}_y}{\partial q_i}\right) \tag{11}$$

$$= \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}_y}{\partial p_x}+\frac{\partial\bar{x}}{\partial y}\frac{\partial\bar{p}_y}{\partial p_y} \tag{12}$$

$$= \sin\theta\cos\theta-\sin\theta\cos\theta \tag{13}$$

$$= 0 \tag{14}$$

Similarly

$$\{\bar{y},\bar{p}_x\} = 0 \tag{15}$$

$$\left\{\bar{y},\bar{p}_y\right\} = 1 \tag{16}$$

Example 2. The transformation from 2-d rectangular to polar coordinates is given by

$$\rho = \sqrt{x^2+y^2} \tag{17}$$

$$\phi = \arctan\frac{y}{x} \tag{18}$$

$$p_\rho = \frac{xp_x+yp_y}{\sqrt{x^2+y^2}} \tag{19}$$

$$p_\phi = xp_y-yp_x \tag{20}$$

For the Poisson brackets we have

$$\{\rho,\phi\} = \sum_i\left(\frac{\partial\rho}{\partial q_i}\frac{\partial\phi}{\partial p_i}-\frac{\partial\rho}{\partial p_i}\frac{\partial\phi}{\partial q_i}\right) \tag{21}$$

$$= 0 \tag{22}$$

because, again, the coordinates don't depend on the momenta.

In this case, however, the new momenta do depend on the old coordinates, so we need to actually do some calculation.

$$\{p_\rho, p_\phi\} = \sum_i \left(\frac{\partial p_\rho}{\partial q_i} \frac{\partial p_\phi}{\partial p_i} - \frac{\partial p_\rho}{\partial p_i} \frac{\partial p_\phi}{\partial q_i} \right) \tag{23}$$

$$= \left(-\frac{x(xp_x + yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_x}{\sqrt{x^2+y^2}} \right)(-y) - \frac{x}{\sqrt{x^2+y^2}} p_y +$$

$$\left(-\frac{y(xp_x + yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_y}{\sqrt{x^2+y^2}} \right) x - \frac{y}{\sqrt{x^2+y^2}} (-p_x) \tag{24}$$

$$= -\frac{y^2(yp_x - xp_y)}{(x^2+y^2)^{3/2}} - \frac{x}{\sqrt{x^2+y^2}} p_y - \frac{x^2(yp_x - xp_y)}{(x^2+y^2)^{3/2}} + \frac{y}{\sqrt{x^2+y^2}} p_x \tag{25}$$

$$= -\frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}} + \frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}} \tag{26}$$

$$= 0 \tag{27}$$

Finally, we need to work out the mixed brackets.

$$\left\{\rho,p_{\rho}\right\} = \sum_{i}\left(\frac{\partial\rho}{\partial q_{i}}\frac{\partial p_{\rho}}{\partial p_{i}}-\frac{\partial\rho}{\partial p_{i}}\frac{\partial p_{\rho}}{\partial q_{i}}\right) \tag{28}$$

$$= \frac{x^{2}}{x^{2}+y^{2}}-0+\frac{y^{2}}{x^{2}+y^{2}}-0 \tag{29}$$

$$= 1 \tag{30}$$

$$\left\{\rho,p_{\phi}\right\} = \sum_{i}\left(\frac{\partial\rho}{\partial q_{i}}\frac{\partial p_{\phi}}{\partial p_{i}}-\frac{\partial\rho}{\partial p_{i}}\frac{\partial p_{\phi}}{\partial q_{i}}\right) \tag{31}$$

$$= -\frac{xy}{\sqrt{x^{2}+y^{2}}}-0+\frac{xy}{\sqrt{x^{2}+y^{2}}}-0 \tag{32}$$

$$= 0 \tag{33}$$

$$\left\{\phi,p_{\rho}\right\} = \sum_{i}\left(\frac{\partial\phi}{\partial q_{i}}\frac{\partial p_{\rho}}{\partial p_{i}}-\frac{\partial\phi}{\partial p_{i}}\frac{\partial p_{\rho}}{\partial q_{i}}\right) \tag{34}$$

$$= -\frac{y}{x\left(1+\frac{y^{2}}{x^{2}}\right)\sqrt{x^{2}+y^{2}}}-0+\frac{y}{x\left(1+\frac{y^{2}}{x^{2}}\right)\sqrt{x^{2}+y^{2}}}-0 \tag{35}$$

$$= 0 \tag{36}$$

$$\left\{\phi,p_{\phi}\right\} = \sum_{i}\left(\frac{\partial\phi}{\partial q_{i}}\frac{\partial p_{\phi}}{\partial p_{i}}-\frac{\partial\phi}{\partial p_{i}}\frac{\partial p_{\phi}}{\partial q_{i}}\right) \tag{37}$$

$$= \frac{y^{2}}{x^{2}\left(1+\frac{y^{2}}{x^{2}}\right)}-0+\frac{1}{1+\frac{y^{2}}{x^{2}}}-0 \tag{38}$$

$$= \frac{y^{2}}{x^{2}+y^{2}}+\frac{x^{2}}{x^{2}+y^{2}} \tag{39}$$

$$= 1 \tag{40}$$

Thus all the Poisson brackets are correct, so the transformation is canonical.

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