CANONICAL TRANSFORMATIONS IN 2-D: ROTATIONS AND POLAR COORDINATES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 2.7.4 - 2.7.5.

Here are a couple of examples of canonical variable transformations.

Example 1. We rotate the 2-d rectangular coordinates through an angle θ, giving the transformations

$$\bar{x} = x\cos\theta - y\sin\theta \tag{1}$$
$$\bar{y} = x\sin\theta + y\cos\theta \tag{2}$$
$$\overline{p}_x = p_x\cos\theta - p_y\sin\theta \tag{3}$$
$$\overline{p}_y = p_x\sin\theta + p_y\cos\theta \tag{4}$$

To show this is a canonical transformation, we must evaluate the Poisson brackets. Here, $q_1 = x$ and $q_2 = y$. Remember that θ is a constant in these derivatives.

$$\{\bar{x},\bar{y}\} = \sum_i \left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{y}}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{y}}{\partial q_i}\right) \tag{5}$$
$$= 0 \tag{6}$$

since neither coordinate depends on any momentum. Similarly $\left\{\overline{p}_x,\overline{p}_y\right\} = 0$ since this Poisson bracket contains derivatives of $\overline{p}_i$ with respect to q_i and these are all zero. The remaining Poisson bracket are of the form $\left\{\overline{q}_i,\overline{p}_j\right\}$. There are four of these, but we'll work out only a couple. The other two have similar forms.

$$\begin{aligned}
\{\bar{x},\bar{p}_x\} &= \sum_i \left(\frac{\partial \bar{x}}{\partial q_i}\frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i}\frac{\partial \bar{p}_x}{\partial q_i}\right) && (7)\\
&= \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{p}_x}{\partial p_x} + \frac{\partial \bar{x}}{\partial y}\frac{\partial \bar{p}_x}{\partial p_y} && (8)\\
&= \cos^2\theta + \sin^2\theta && (9)\\
&= 1 && (10)\\
\{\bar{x},\bar{p}_y\} &= \sum_i \left(\frac{\partial \bar{x}}{\partial q_i}\frac{\partial \bar{p}_y}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i}\frac{\partial \bar{p}_y}{\partial q_i}\right) && (11)\\
&= \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{p}_y}{\partial p_x} + \frac{\partial \bar{x}}{\partial y}\frac{\partial \bar{p}_y}{\partial p_y} && (12)\\
&= \sin\theta\cos\theta - \sin\theta\cos\theta && (13)\\
&= 0 && (14)
\end{aligned}$$

Similarly

$$\begin{aligned}
\{\bar{y},\bar{p}_x\} &= 0 && (15)\\
\{\bar{y},\bar{p}_y\} &= 1 && (16)
\end{aligned}$$

Example 2. The transformation from 2-d rectangular to polar coordinates is given by

$$\begin{aligned}
\rho &= \sqrt{x^2+y^2} && (17)\\
\phi &= \arctan\frac{y}{x} && (18)\\
p_\rho &= \frac{xp_x+yp_y}{\sqrt{x^2+y^2}} && (19)\\
p_\phi &= xp_y - yp_x && (20)
\end{aligned}$$

For the Poisson brackets we have

$$\begin{aligned}
\{\rho,\phi\} &= \sum_i \left(\frac{\partial \rho}{\partial q_i}\frac{\partial \phi}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial \phi}{\partial q_i}\right) && (21)\\
&= 0 && (22)
\end{aligned}$$

because, again, the coordinates don't depend on the momenta.

In this case, however, the new momenta do depend on the old coordinates, so we need to actually do some calculation.

$$
\begin{aligned}
\{p_\rho, p_\phi\} &= \sum_i \left(\frac{\partial p_\rho}{\partial q_i}\frac{\partial p_\phi}{\partial p_i} - \frac{\partial p_\rho}{\partial p_i}\frac{\partial p_\phi}{\partial q_i} \right) && (23)\\
&= \left(-\frac{x(xp_x+yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_x}{\sqrt{x^2+y^2}} \right)(-y) - \frac{x}{\sqrt{x^2+y^2}} p_y + \\
&\quad \left(-\frac{y(xp_x+yp_y)}{(x^2+y^2)^{3/2}} + \frac{p_y}{\sqrt{x^2+y^2}} \right) x - \frac{y}{\sqrt{x^2+y^2}}(-p_x) && (24)\\
&= -\frac{y^2(yp_x-xp_y)}{(x^2+y^2)^{3/2}} - \frac{x}{\sqrt{x^2+y^2}} p_y - \frac{x^2(yp_x-xp_y)}{(x^2+y^2)^{3/2}} + \frac{y}{\sqrt{x^2+y^2}} p_x && (25)\\
&= -\frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}} + \frac{y^3 p_x + x^3 p_y}{(x^2+y^2)^{3/2}} && (26)\\
&= 0 && (27)
\end{aligned}
$$

Finally, we need to work out the mixed brackets.

$$\left\{\rho,p_{\rho}\right\} = \sum_{i}\left(\frac{\partial\rho}{\partial q_{i}}\frac{\partial p_{\rho}}{\partial p_{i}}-\frac{\partial\rho}{\partial p_{i}}\frac{\partial p_{\rho}}{\partial q_{i}}\right) \tag{28}$$

$$= \frac{x^{2}}{x^{2}+y^{2}}-0+\frac{y^{2}}{x^{2}+y^{2}}-0 \tag{29}$$

$$= 1 \tag{30}$$

$$\left\{\rho,p_{\phi}\right\} = \sum_{i}\left(\frac{\partial\rho}{\partial q_{i}}\frac{\partial p_{\phi}}{\partial p_{i}}-\frac{\partial\rho}{\partial p_{i}}\frac{\partial p_{\phi}}{\partial q_{i}}\right) \tag{31}$$

$$= -\frac{xy}{\sqrt{x^{2}+y^{2}}}-0+\frac{xy}{\sqrt{x^{2}+y^{2}}}-0 \tag{32}$$

$$= 0 \tag{33}$$

$$\left\{\phi,p_{\rho}\right\} = \sum_{i}\left(\frac{\partial\phi}{\partial q_{i}}\frac{\partial p_{\rho}}{\partial p_{i}}-\frac{\partial\phi}{\partial p_{i}}\frac{\partial p_{\rho}}{\partial q_{i}}\right) \tag{34}$$

$$= -\frac{y}{x\left(1+\frac{y^{2}}{x^{2}}\right)\sqrt{x^{2}+y^{2}}}-0+\frac{y}{x\left(1+\frac{y^{2}}{x^{2}}\right)\sqrt{x^{2}+y^{2}}}-0 \tag{35}$$

$$= 0 \tag{36}$$

$$\left\{\phi,p_{\phi}\right\} = \sum_{i}\left(\frac{\partial\phi}{\partial q_{i}}\frac{\partial p_{\phi}}{\partial p_{i}}-\frac{\partial\phi}{\partial p_{i}}\frac{\partial p_{\phi}}{\partial q_{i}}\right) \tag{37}$$

$$= \frac{y^{2}}{x^{2}\left(1+\frac{y^{2}}{x^{2}}\right)}-0+\frac{1}{1+\frac{y^{2}}{x^{2}}}-0 \tag{38}$$

$$= \frac{y^{2}}{x^{2}+y^{2}}+\frac{x^{2}}{x^{2}+y^{2}} \tag{39}$$

$$= 1 \tag{40}$$

Thus all the Poisson brackets are correct, so the transformation is canonical.

PINGBACKS

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