CANONICAL TRANSFORMATIONS: A FEW MORE EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercises 02.07.06 - 02.07.07, 02.07.08(4).

Here are a few more examples of canonical variable transformations.

Example 1. First, we revisit the two-body problem, in which we simplified the problem by transforming from the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the masses m_1 and m_2 to two new position vectors:

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 \tag{1}$$

$$\mathbf{r}_{CM} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} \tag{2}$$

Here $M \equiv m_1 + m_2$ is the total mass, $\mathbf{r}$ is the relative position, and $\mathbf{r}_{CM}$ is the position of the centre of mass. The conjugate momenta in the original system are

$$\mathbf{p}_i = m\dot{\mathbf{r}}_i \tag{3}$$

The conjugate momenta transform according to

$$\mathbf{p}_{CM} = M\mathbf{r}_{CM} = \mathbf{p}_1 + \mathbf{p}_2 \tag{4}$$

$$\mathbf{p} = \mu\dot{\mathbf{r}} \tag{5}$$

$$= \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{M} \tag{6}$$

where $\mu = m_1 m_2/M$ is the reduced mass.

To check that this is a canonical transformation, we need to calculate the Poisson brackets. To make things easier, note that the new coordinates depend only on the old coordinates (and not on the momenta), and conversely, the new momenta depend only on the old momenta (and not on the coordinates). Since the Poisson brackets $\left\{\overline{q}_i,\overline{q}_j\right\}$and $\left\{\overline{p}_i,\overline{p}_j\right\}$all involve taking derivatives of coordinates with respect to momenta (in the first case) or momenta with respect to coordinates (in the second case), all these brackets are zero. We need, therefore, to check only the mixed brackets between coordinates and momenta.

Because we're dealing with 3-d vector equations, there are 3 components to each vector and to be thorough, we need to calculate all possible brackets between all pairs of components. However, if we do the x component of each, it should be obvious that the y and z components behave in the same way.

First, consider

$$\{r_x, p_x\} = \sum_i \left(\frac{\partial r_x}{\partial q_i} \frac{\partial p_x}{\partial p_i} - \frac{\partial r_x}{\partial p_i} \frac{\partial p_x}{\partial q_i} \right) \tag{7}$$

In the RHS, the term q_i stands for all 6 components of the original position vectors, that is $q_i = \left\{ r_{1x}, r_{1y}, \ldots, r_{2z} \right\}$ and the term p_i in the denominators refers to all 6 components of the original momentum vectors. The p_x in the numerators refers to the x component of **p** in 6. Hopefully this won't cause too much confusion.

The second term on the RHS is zero because it involves derivatives of coordinates with respect to momenta (and vice versa). In the first term, r_x depends only the x components of $\mathbf{r}_1$ and $\mathbf{r}_2$, and p_x depends only on the x components of $\mathbf{p}_1$and $\mathbf{p}_2$, so we have

$$\begin{aligned}
\{r_x, p_x\} &= \frac{\partial r_x}{\partial r_{1x}} \frac{\partial p_x}{\partial p_{1x}} + \frac{\partial r_x}{\partial r_{2x}} \frac{\partial p_x}{\partial p_{2x}} & (8)\\
&= (1) \frac{m_2}{M} + (-1) \left(-\frac{m_1}{M} \right) & (9)\\
&= \frac{m_1 + m_2}{M} & (10)\\
&= 1 & (11)
\end{aligned}$$

The same result is obtained for the y and z components. If we look at mixing two different components, we have, for example

$$\{r_x, p_y\} = \frac{\partial r_x}{\partial r_{1x}} \frac{\partial p_y}{\partial p_{1x}} + \frac{\partial r_x}{\partial r_{2x}} \frac{\partial p_y}{\partial p_{2x}} + \frac{\partial r_x}{\partial r_{1y}} \frac{\partial p_y}{\partial p_{1y}} + \frac{\partial r_x}{\partial r_{2y}} \frac{\partial p_y}{\partial p_{2y}} = 0 \tag{12}$$

This is zero because each term in the sum contains a derivative of an x component with respect to a y component (or vice versa), all of which are zero.

For the centre of mass components, we have

$$\{r_{CMx}, p_{CMx}\} = \frac{\partial r_{CMx}}{\partial r_{1x}}\frac{\partial p_{CMx}}{\partial p_{1x}} + \frac{\partial r_{CMx}}{\partial r_{2x}}\frac{\partial p_{CMx}}{\partial p_{2x}} \tag{13}$$

$$= \frac{m_1}{M}(1) + \frac{m_2}{M}(1) \tag{14}$$

$$= 1 \tag{15}$$

$$\{r_{CMx}, p_{CMy}\} = \frac{\partial r_{CMx}}{\partial r_{1x}}\frac{\partial p_{CMy}}{\partial p_{1x}} + \frac{\partial r_{CMx}}{\partial r_{2x}}\frac{\partial p_{CMy}}{\partial p_{2x}} + \frac{\partial r_{CMx}}{\partial r_{1y}}\frac{\partial p_{CMy}}{\partial p_{1y}} + \frac{\partial r_{CMx}}{\partial r_{2y}}\frac{\partial p_{CMy}}{\partial p_{2y}} \tag{16}$$

$$= 0 \tag{17}$$

where the last bracket is zero for the same reason as $\{r_x, p_y\}$: we're mixing x and y in the derivatives. Again, it should be obvious that the brackets for the other combinations of x, y and z components work out the same way. We can also verify that the Poisson brackets between relative and centre of mass coordinates are zero by the same method. That is

$$\{r_{CMi}, p_j\} = \{r_i, p_{CMj}\} = 0 \tag{18}$$

where i and j take on the values x, y and z.

Example 2. A bizarre transformation of variables in one dimension is given by

$$\bar{q} = \ln\frac{\sin p}{q} = \ln\sin p - \ln q \tag{19}$$

$$\bar{p} = q\cot p \tag{20}$$

To show this is canonical, we need calculate only $\{\bar{q}, \bar{p}\}$ (since the Poisson bracket of a function with itself is always zero, we have $\{\bar{q}, \bar{q}\} = \{\bar{p}, \bar{p}\} = 0$). We need one rather obscure derivative of a trig function.

$$\frac{d}{dp}\cot p = \frac{d}{dp}\left(\frac{\cos p}{\sin p}\right) \tag{21}$$

$$= \frac{-\sin^2 p - \cos^2 p}{\sin^2 p} \tag{22}$$

$$= -1 - \cot^2 p \tag{23}$$

We get

$$\begin{aligned}
\{\overline{q},\overline{p}\} &= \frac{\partial\overline{q}}{\partial q}\frac{\partial\overline{p}}{\partial p}-\frac{\partial\overline{q}}{\partial p}\frac{\partial\overline{p}}{\partial q} &(24)\\
&= \left(-\frac{1}{q}\right)\left(q\left(-1-\cot^2 p\right)\right)-\frac{\cos p}{\sin p}\cot p &(25)\\
&= 1+\cot^2 p-\cot^2 p &(26)\\
&= 1 &(27)
\end{aligned}$$

Thus the transformation is canonical.

Example 3. Finally, we return to the point transformation, which is given in general by

$$\begin{aligned}
\overline{q}_i &= \overline{q}_i\left(q_1,\ldots,q_n\right) &(28)\\
\overline{p}_i &= \sum_j \frac{\partial q_j}{\partial\overline{q}_i}p_j &(29)
\end{aligned}$$

In this case, the coordinate transformation to $\overline{q}$ is completely arbitrary, but the momentum transformation must follow the formula given. The derivatives $\frac{\partial q_i}{\partial\overline{q}_j}$ in the formula for $\overline{p}_i$ are taken at constant $\overline{q}$. As in the earlier examples, since the coordinate formulas depend only on the old coordinates, and the momentum formulas depend only on the old momenta, the Poisson brackets satisfy

$$\left\{\overline{q}_i,\overline{q}_j\right\}=\left\{\overline{p}_i,\overline{p}_j\right\}=0 \tag{30}$$

For the mixed brackets, we have

$$\begin{aligned}
\left\{\overline{q}_i,\overline{p}_j\right\} &= \sum_k\left(\frac{\partial\overline{q}_i}{\partial q_k}\frac{\partial\overline{p}_j}{\partial p_k}-\frac{\partial\overline{q}_i}{\partial p_k}\frac{\partial\overline{p}_j}{\partial q_k}\right) &(31)\\
&= \sum_k\frac{\partial\overline{q}_i}{\partial q_k}\frac{\partial q_k}{\partial\overline{q}_j} &(32)\\
&= \frac{\partial\overline{q}_i}{\partial\overline{q}_j} &(33)\\
&= \delta_{ij} &(34)
\end{aligned}$$

The second term in the first line is zero (mixed derivatives again). We used 29 to calculate the derivative $\frac{\partial\overline{p}_j}{\partial p_k}$ and get the second line and then notice that the sum is an expansion of the chain rule for the derivative in line 3. Since $\overline{q}_i$ and $\overline{q}_j$ are independent variables, the result is that given in the last line. Thus a point transformation is a canonical transformation.