POISSON BRACKETS ARE INVARIANT UNDER A CANONICAL TRANSFORMATION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercise 2.7.9.

The Poisson bracket of two functions is defined as

$$\{\omega,\sigma\}=\sum_{i}\left(\frac{\partial\omega}{\partial q_{i}}\frac{\partial\sigma}{\partial p_{i}}-\frac{\partial\omega}{\partial p_{i}}\frac{\partial\sigma}{\partial q_{i}}\right) \tag{1}$$

Calculating the Poisson bracket requires knowing ω and σ as functions of the coordinates q_i and momenta p_i in the particular coordinate system we're using. However, we've seen that the Euler-Lagrange and Hamilton's equations are invariant under a canonical transformation and since the Poisson bracket is a fundamental quantity in classical mechanics, in particular because the time derivative of a function ω is the Poisson bracket $\{\omega,H\}$ with the Hamiltonian, it's natural to ask how the Poisson bracket of two functions transforms under a canonical transformation.

The simplest way of finding out (although not the most elegant) is to write the canonical transformation as

$$\bar{q}_{i} = \bar{q}_{i}(q,p) \tag{2}$$

$$\bar{p}_{i} = \bar{p}(q,p) \tag{3}$$

We can then write the Poisson bracket in the new coordinates as

$$\{\omega,\sigma\}_{\bar{q},\bar{p}}=\sum_{j}\left(\frac{\partial\omega}{\partial\bar{q}_{j}}\frac{\partial\sigma}{\partial\bar{p}_{j}}-\frac{\partial\omega}{\partial\bar{p}_{j}}\frac{\partial\sigma}{\partial\bar{q}_{j}}\right) \tag{4}$$

Assuming the transformation is invertible, we can use the chain rule to calculate the derivatives with respect to the barred coordinates. This gives the following (we've used the summation convention in which any index repeated twice in a product is summed; thus in the following, there are implied sums over i, j and k):

$$\begin{aligned}
\{\omega,\sigma\}_{\bar{q},\bar{p}} &= \left(\frac{\partial\omega}{\partial q_i}\frac{\partial q_i}{\partial\bar{q}_j}+\frac{\partial\omega}{\partial p_i}\frac{\partial p_i}{\partial\bar{q}_j}\right)\left(\frac{\partial\sigma}{\partial q_k}\frac{\partial q_k}{\partial\bar{p}_j}+\frac{\partial\sigma}{\partial p_k}\frac{\partial p_k}{\partial\bar{p}_j}\right)- \\
&\quad \left(\frac{\partial\omega}{\partial q_i}\frac{\partial q_i}{\partial\bar{p}_j}+\frac{\partial\omega}{\partial p_i}\frac{\partial p_i}{\partial\bar{p}_j}\right)\left(\frac{\partial\sigma}{\partial q_k}\frac{\partial q_k}{\partial\bar{q}_j}+\frac{\partial\sigma}{\partial p_k}\frac{\partial p_k}{\partial\bar{q}_j}\right) \qquad (5)\\
&= \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_k}\left(\frac{\partial q_i}{\partial\bar{q}_j}\frac{\partial p_k}{\partial\bar{p}_j}-\frac{\partial q_i}{\partial\bar{p}_j}\frac{\partial p_k}{\partial\bar{q}_j}\right)+ \\
&\quad \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial q_k}\left(\frac{\partial p_i}{\partial\bar{q}_j}\frac{\partial q_k}{\partial\bar{p}_j}-\frac{\partial p_i}{\partial\bar{p}_j}\frac{\partial q_k}{\partial\bar{q}_j}\right)+ \\
&\quad \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial q_k}\left(\frac{\partial q_i}{\partial\bar{q}_j}\frac{\partial q_k}{\partial\bar{p}_j}-\frac{\partial q_i}{\partial\bar{p}_j}\frac{\partial q_k}{\partial\bar{q}_j}\right)+ \\
&\quad \frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_k}\left(\frac{\partial p_i}{\partial\bar{q}_j}\frac{\partial p_k}{\partial\bar{p}_j}-\frac{\partial p_i}{\partial\bar{p}_j}\frac{\partial p_k}{\partial\bar{q}_j}\right) \qquad (6)\\
&= \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_k}\{q_i,p_k\}+\frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial q_k}\{p_i,q_k\}+ \\
&\quad \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial q_k}\{q_i,q_k\}+\frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_k}\{p_i,p_k\} \qquad (7)
\end{aligned}$$

For a canonical transformation, the Poisson brackets in the last equation satisfy

$$\{q_i,p_k\} = -\{p_i,q_k\}=\delta_{ik} \qquad (8)$$
$$\{q_i,q_k\} = \{p_i,p_k\}=0 \qquad (9)$$

[Actually, we had worked out these conditions for the barred coordinates in terms of the original coordinates, but since the transformation is invertible and both sets of coordinates are canonical, the Poisson brackets work either way.] Applying these conditions to the above, we find

$$\{\omega,\sigma\}_{\bar{q},\bar{p}} = \left(\frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_k}-\frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_k}\right)\delta_{ik} \qquad (10)$$
$$= \frac{\partial\omega}{\partial q_i}\frac{\partial\sigma}{\partial p_i}-\frac{\partial\omega}{\partial p_i}\frac{\partial\sigma}{\partial p_i} \qquad (11)$$
$$= \{\omega,\sigma\}_{q,p} \qquad (12)$$

Thus the Poisson bracket is invariant under a canonical transformation.