PASSIVE, REGULAR AND ACTIVE TRANSFORMATIONS. INVARIANCE OF THE HAMILTONIAN AND GENERATORS OF TRANSFORMATIONS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Sections 2.7 & 2.8; Exercises 2.8.1 - 2.8.2.

The canonical transformations we've considered so far are of the form

$$\overline{q}_i = \overline{q}_i(q,p) \tag{1}$$
$$\overline{p}_i = \overline{p}_i(q,p) \tag{2}$$

The interpretation of these transformations is that we are using a new set of coordinates and momenta to describe the *same* point in phase space. For example, in 2-d we can describe the point one unit along the y axis by the coordinates $x=0, y=1$ if we use rectangular coordinates, or by $r=1, \theta=\frac{\pi}{2}$ if we use polar coordinates. The numerical values of the coordinates are different in the two systems, but the geometric point being described is the same. Such a transformation is called a *passive transformation*. In a passive transformation, any function ω always has the same value at a given point in phase space no matter which coordinate system we're using, so we can say that

$$\omega(q,p) = \omega(\bar{q},\bar{p}) \tag{3}$$

where it is understood that (q,p) and $(\bar{q},\bar{p})$ both refer to the same point, but in different representations.

One characteristic of a passive transformation is that the ranges of the variables used to represent a point in phase space need not be the same in the two systems. For example, in 2-d rectangular coordinates, both x and y can range from $-\infty$ to $+\infty$, while in polar coordinates r ranges between 0 and $+\infty$ while the angle θ runs between 0 and 2π.

A special type of transformation is a *regular transformation*, in which the variables in the two systems have the same ranges. For example, if we translate a 2-d system by 1 unit along the x axis, the new coordinates are related to the old ones by

$$\begin{aligned}\bar{x} &= x-1 &(4)\\ \bar{y} &= y &(5)\end{aligned}$$

Both the original and barred systems have the same range ($-\infty$ to $+\infty$).

Although we can interpret a regular transformation as a passive transformation, we can also think of it in a different way. We can image that instead of just providing a different label for the same point that the transformed coordinate has actually shifted the system to a new location in phase space. In the above example, this would mean that we have physically moved the system by 1 unit along the x axis. This interpretation is known as an *active transformation*.

If a function ω is invariant under an active transformation, then it satisfies the condition

$$\omega(q,p) = \omega(\bar{q},\bar{p}) \tag{6}$$

Although mathematically this is the same as 3, physically it means something quite different, since now the points (q,p) and $(\bar{q},\bar{p})$ refer to *different* points in phase space, so we're saying that the function ω does not change when we move the physical system in the way specified by the active transformation.

We now restrict ourselves to talking about regular canonical transformations. Consider some dynamical variable (it could be momentum or angular momentum, for example) $g(q,p)$ and suppose we define the transformations

$$\begin{aligned}\bar{q}_i &= q_i+\varepsilon\frac{\partial g}{\partial p_i}\equiv q_i+\delta q_i &(7)\\ \bar{p}_i &= p_i-\varepsilon\frac{\partial g}{\partial q_i}\equiv p_i+\delta p_i &(8)\end{aligned}$$

where ε is some infinitesimal quantity.

First, we need to show that, to first order in ε, this is a canonical transformation. The required conditions for this are

$$\begin{aligned}\left\{\overline{q}_i,\overline{q}_j\right\} &= \left\{\overline{p}_i,\overline{p}_j\right\}=0 &(9)\\ \left\{\overline{q}_i,\overline{p}_j\right\} &= \delta_{ij} &(10)\end{aligned}$$

Consider first (we'll use the summation convention, so the index k is summed in what follows):

$$
\begin{aligned}
\left\{\overline{q}_i, \overline{p}_j\right\} &= \frac{\partial}{\partial q_k}\left(q_i + \varepsilon\frac{\partial g}{\partial p_i}\right)\frac{\partial}{\partial p_k}\left(p_j - \varepsilon\frac{\partial g}{\partial q_j}\right) - \\
&\quad \frac{\partial}{\partial p_k}\left(q_i + \varepsilon\frac{\partial g}{\partial p_i}\right)\frac{\partial}{\partial q_k}\left(p_j - \varepsilon\frac{\partial g}{\partial q_j}\right) && (11)\\
&= \left(\delta_{ik} + \varepsilon\frac{\partial^2 g}{\partial p_i q_k}\right)\left(\delta_{jk} - \varepsilon\frac{\partial^2 g}{\partial p_k q_j}\right) - \\
&\quad \left(0 + \varepsilon\frac{\partial^2 g}{\partial p_i p_k}\right)\left(0 - \varepsilon\frac{\partial^2 g}{\partial q_j q_k}\right) && (12)
\end{aligned}
$$

The zeroes in the last line follow from the fact that q_k and p_k are independent variables. We can now keep terms only up to first order in ε to get

$$
\begin{aligned}
\left\{\overline{q}_i, \overline{p}_j\right\} &= \delta_{ik}\delta_{jk} + \varepsilon\left(\frac{\partial^2 g}{\partial p_i q_k}\delta_{jk} - \frac{\partial^2 g}{\partial p_k q_j}\delta_{ik}\right) && (13)\\
&= \delta_{ij} + \varepsilon\left(\frac{\partial^2 g}{\partial p_i q_j} - \frac{\partial^2 g}{\partial p_i q_j}\right) && (14)\\
&= \delta_{ij} && (15)
\end{aligned}
$$

The other two brackets work out similarly:

$$
\begin{aligned}
\left\{\overline{q}_i, \overline{q}_j\right\} &= \frac{\partial}{\partial q_k}\left(q_i + \varepsilon\frac{\partial g}{\partial p_i}\right)\frac{\partial}{\partial p_k}\left(q_j + \varepsilon\frac{\partial g}{\partial p_j}\right) - \\
&\quad \frac{\partial}{\partial p_k}\left(q_i + \varepsilon\frac{\partial g}{\partial p_i}\right)\frac{\partial}{\partial q_k}\left(q_j + \varepsilon\frac{\partial g}{\partial p_j}\right) && (16)\\
&= \left(\delta_{ik} + \varepsilon\frac{\partial^2 g}{\partial p_i q_k}\right)\left(0 + \varepsilon\frac{\partial^2 g}{\partial p_k p_j}\right) - \\
&\quad \left(0 + \varepsilon\frac{\partial^2 g}{\partial p_i p_k}\right)\left(\delta_{jk} + \varepsilon\frac{\partial^2 g}{\partial p_j q_k}\right) && (17)\\
&= \delta_{ik}\varepsilon\frac{\partial^2 g}{\partial p_k p_j} - \delta_{jk}\varepsilon\frac{\partial^2 g}{\partial p_i p_k} && (18)\\
&= \varepsilon\left(\frac{\partial^2 g}{\partial p_i p_j} - \frac{\partial^2 g}{\partial p_i p_j}\right) && (19)\\
&= 0 && (20)
\end{aligned}
$$

$$\begin{aligned}
\{\overline{p}_i,\overline{p}_j\} &= \frac{\partial}{\partial q_k}\left(p_i-\varepsilon\frac{\partial g}{\partial q_i}\right)\frac{\partial}{\partial p_k}\left(p_j-\varepsilon\frac{\partial g}{\partial q_j}\right)- \\
&\quad \frac{\partial}{\partial p_k}\left(p_i-\varepsilon\frac{\partial g}{\partial q_i}\right)\frac{\partial}{\partial q_k}\left(p_j-\varepsilon\frac{\partial g}{\partial q_j}\right) && (21)\\
&= \left(0-\varepsilon\frac{\partial^2 g}{\partial q_i q_k}\right)\left(\delta_{jk}-\varepsilon\frac{\partial^2 g}{\partial p_k q_j}\right)- \\
&\quad \left(\delta_{ik}-\varepsilon\frac{\partial^2 g}{\partial q_i p_k}\right)\left(0-\varepsilon\frac{\partial^2 g}{\partial q_j q_k}\right) && (22)\\
&= -\delta_{jk}\varepsilon\frac{\partial^2 g}{\partial q_i q_k}+\delta_{ik}\varepsilon\frac{\partial^2 g}{\partial q_k q_j} && (23)\\
&= -\varepsilon\left(\frac{\partial^2 g}{\partial q_i q_j}-\frac{\partial^2 g}{\partial q_i q_j}\right) && (24)\\
&= 0 && (25)
\end{aligned}$$

Thus all the brackets check out, so the transformation is canonical.

The point of all this is that, if the Hamiltonian is invariant under the transformations 7 and 8 then the variable g is conserved (that is, doesn't change with time). g is called the *generator* of the transformation. We can verify this by using the chain rule to calculate the variation in H:

$$\begin{aligned}
\delta H &= \frac{\partial H}{\partial q_i}\delta q_i+\frac{\partial H}{\partial p_i}\delta p_i && (26)\\
&= \varepsilon\left[\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i}\right] && (27)\\
&= \varepsilon\{H,g\} && (28)
\end{aligned}$$

Since H is invariant, we must have $\delta H=0$, so

$$\{H,g\}=0 \tag{29}$$

However, this is the condition for g to be conserved. QED.

Example. Suppose we have a two particle system moving in one dimension, with positions q_1,q_2 and momenta p_1,p_2. If we take

$$g=p_1+p_2 \tag{30}$$

we get

$$\delta q_i = \epsilon \frac{\partial g}{\partial p_i} = \epsilon \tag{31}$$

$$\delta p_i = -\epsilon \frac{\partial g}{\partial q_i} = 0 \tag{32}$$

That is, each particle gets shifted by the same amount ϵ but the momentum of each particle remains unchanged. Thus the total momentum is the generator of infinitesimal translations. The physical interpretation of this is that, since the momentum of each particle is conserved, the total kinetic energy

$$T = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \tag{33}$$

remains unchanged. Since the total energy is invariant, the total potential energy of the system is unaffected by a translation, which means that there is no external force on the system.

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