INFINITESIMAL ROTATIONS IN CANONICAL AND NONCANonical TRANFORMATIONS

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Here are a couple of examples of transformations of variables and their consequences with regard to conservation laws.

First, we look at the 2-d harmonic oscillator where the Hamiltonian is

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2} m \omega^2 (x^2 + y^2)$$  \hspace{1cm} (1)

If we rotate the system so that both the coordinates and momenta get rotated, then

$$\bar{x} = x \cos \theta - y \sin \theta$$  \hspace{1cm} (2)

$$\bar{y} = x \sin \theta + y \cos \theta$$  \hspace{1cm} (3)

$$\bar{p}_x = p_x \cos \theta - p_y \sin \theta$$  \hspace{1cm} (4)

$$\bar{p}_y = p_x \sin \theta + p_y \cos \theta$$  \hspace{1cm} (5)

We can show by direct calculation that $H$ is invariant under this transformation, and we can verify that this is a canonical transformation. Shankar shows in his equation 2.8.8 that the generator of this transformation is the angular momentum $\ell_z = xp_y - yp_x$.

However, if we rotate only the coordinates and not the momenta, we get the transformation:

$$\bar{x} = x \cos \theta - y \sin \theta$$  \hspace{1cm} (6)

$$\bar{y} = x \sin \theta + y \cos \theta$$  \hspace{1cm} (7)

$$\bar{p}_x = p_x$$  \hspace{1cm} (8)

$$\bar{p}_y = p_y$$  \hspace{1cm} (9)

Again, we can show by direct calculation that

$$\bar{x}^2 + \bar{y}^2 = x^2 + y^2$$  \hspace{1cm} (10)
so \( H \) is also invariant under this transformation. However, this transformation is noncanonical, as we can see by calculating one of the Poisson brackets:

\[
\{ \bar{x}, \bar{p}_x \} = \sum_i \left( \frac{\partial \bar{p}_x}{\partial q_i} \frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{p}_x}{\partial p_i} \frac{\partial \bar{p}_x}{\partial q_i} \right) = \cos \theta \neq 1
\]

The other mixed brackets (with a coordinate and a momentum) are also not either 0 or 1 as would be required if the transformation were to be canonical.

In order for this transformation to give rise to a conservation law, we would need to find a generator \( g \) that satisfied, for an infinitesimal rotation \( \varepsilon \):

\[
\bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i
\]

\[
\bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i
\]

For an infinitesimal rotation, the transformation becomes

\[
\begin{align*}
\bar{x} &= x - \varepsilon y \\
\bar{y} &= y + \varepsilon x \\
\bar{p}_x &= p_x \\
\bar{p}_y &= p_y
\end{align*}
\]

Therefore, the generator would have to satisfy

\[
\begin{align*}
\frac{\partial g}{\partial p_x} &= -y \\
\frac{\partial g}{\partial p_y} &= x \\
\frac{\partial g}{\partial x} &= 0 \\
\frac{\partial g}{\partial y} &= 0
\end{align*}
\]

The last two conditions state that \( g \) cannot depend on \( x \) or \( y \), but integrating the first two conditions, we get

\[
g = -yp_x + xp_y + f(x, y)
\]
where \( f \) is a function that depends only on \( x \) and/or \( y \). Thus there is no \( g \) that satisfies all four conditions, so there is no conservation law associated with a rotation of the coordinates only, even though the Hamiltonian is invariant under this transformation. Only canonical transformations that leave \( H \) invariant give rise to conservation laws.

As another example, suppose he have the one-dimensional system with

\[
H = \frac{1}{2} \left( p^2 + x^2 \right)
\]

and perform a rotation in phase space, that is, in the \( x - p \) plane:

\[
\begin{align*}
\bar{x} &= x \cos \theta - p \sin \theta \\
\bar{p} &= x \sin \theta + p \cos \theta
\end{align*}
\]

The Hamiltonian is invariant:

\[
\bar{p}^2 + \bar{x}^2 = x^2 \sin^2 \theta + 2xp \sin \theta \cos \theta + p^2 \cos^2 \theta + x^2 \cos^2 \theta - 2xp \sin \theta \cos \theta + p^2 \sin^2 \theta = x^2 + p^2
\]

The transformation is canonical as we can verify by calculating the Poisson bracket

\[
\{ \bar{x}, \bar{p} \} = \frac{\partial x}{\partial \bar{x}} \frac{\partial \bar{p}}{\partial p} - \frac{\partial x}{\partial \bar{p}} \frac{\partial \bar{p}}{\partial \bar{x}} = \cos^2 \theta - ( - \sin^2 \theta ) = 1
\]

An infinitesimal rotation gives the transformation

\[
\begin{align*}
\bar{x} &= x - \varepsilon p \\
\bar{p} &= p + \varepsilon x
\end{align*}
\]

To find the generator, we need to solve [13] and [14]

\[
\begin{align*}
\frac{\partial g}{\partial \bar{p}} &= -p \\
\frac{\partial g}{\partial \bar{x}} &= -x
\end{align*}
\]

These can be integrated to give
\[ g(x,p) = -\frac{1}{2} (p^2 + x^2) + C \] (37)

where \( C \) is a constant of integration. Thus the quantity that is conserved is (apart from the minus sign, which we could eliminate by rotating through \(-\theta\) instead of \(\theta\)) just the original Hamiltonian, or total energy.

PINGBACKS

Pingback: Hamilton’s equations of motion under a regular canonical transformation