

INFINITESIMAL ROTATIONS IN CANONICAL AND NONCANONICAL TRANSFORMATIONS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercises 2.8.3 - 2.8.4.

Here are a couple of examples of transformations of variables and their consequences with regard to conservation laws.

First, we look at the 2-d harmonic oscillator where the Hamiltonian is

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2} m \omega^2 (x^2 + y^2) \quad (1)$$

If we rotate the system so that both the coordinates and momenta get rotated, then

$$\bar{x} = x \cos \theta - y \sin \theta \quad (2)$$

$$\bar{y} = x \sin \theta + y \cos \theta \quad (3)$$

$$\bar{p}_x = p_x \cos \theta - p_y \sin \theta \quad (4)$$

$$\bar{p}_y = p_x \sin \theta + p_y \cos \theta \quad (5)$$

We can show by direct calculation that H is invariant under this transformation, and we can verify that this is a canonical transformation. Shankar shows in his equation 2.8.8 that the generator of this transformation is the angular momentum $\ell_z = xp_y - yp_x$.

However, if we rotate only the coordinates and not the momenta, we get the transformation:

$$\bar{x} = x \cos \theta - y \sin \theta \quad (6)$$

$$\bar{y} = x \sin \theta + y \cos \theta \quad (7)$$

$$\bar{p}_x = p_x \quad (8)$$

$$\bar{p}_y = p_y \quad (9)$$

Again, we can show by direct calculation that

$$\bar{x}^2 + \bar{y}^2 = x^2 + y^2 \quad (10)$$

so H is also invariant under this transformation. However, this transformation is noncanonical, as we can see by calculating one of the Poisson brackets:

$$\{\bar{x}, \bar{p}_x\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_x}{\partial q_i} \right) \quad (11)$$

$$= \cos \theta \neq 1 \quad (12)$$

The other mixed brackets (with a coordinate and a momentum) are also not either 0 or 1 as would be required if the transformation were to be canonical.

In order for this transformation to give rise to a conservation law, we would need to find a generator g that satisfied, for an infinitesimal rotation ε :

$$\bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i \quad (13)$$

$$\bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i \quad (14)$$

For an infinitesimal rotation, the transformation 6 becomes

$$\bar{x} = x - \varepsilon y \quad (15)$$

$$\bar{y} = y + \varepsilon x \quad (16)$$

$$\bar{p}_x = p_x \quad (17)$$

$$\bar{p}_y = p_y \quad (18)$$

Therefore, the generator would have to satisfy

$$\frac{\partial g}{\partial p_x} = -y \quad (19)$$

$$\frac{\partial g}{\partial p_y} = x \quad (20)$$

$$\frac{\partial g}{\partial x} = 0 \quad (21)$$

$$\frac{\partial g}{\partial y} = 0 \quad (22)$$

The last two conditions state that g cannot depend on x or y , but integrating the first two conditions, we get

$$g = -yp_x + xp_y + f(x, y) \quad (23)$$

where f is a function that depends only on x and/or y . Thus there is no g that satisfies all four conditions, so there is no conservation law associated with a rotation of the coordinates only, even though the Hamiltonian is invariant under this transformation. Only canonical transformations that leave H invariant give rise to conservation laws.

As another example, suppose we have the one-dimensional system with

$$H = \frac{1}{2}(p^2 + x^2) \quad (24)$$

and perform a rotation in phase space, that is, in the $x - p$ plane:

$$\bar{x} = x \cos \theta - p \sin \theta \quad (25)$$

$$\bar{p} = x \sin \theta + p \cos \theta \quad (26)$$

The Hamiltonian is invariant:

$$\bar{p}^2 + \bar{x}^2 = x^2 \sin^2 \theta + 2xp \sin \theta \cos \theta + p^2 \cos^2 \theta + \quad (27)$$

$$x^2 \cos^2 \theta - 2xp \sin \theta \cos \theta + p^2 \sin^2 \theta \quad (28)$$

$$= x^2 + p^2 \quad (29)$$

The transformation is canonical as we can verify by calculating the Poisson bracket

$$\{\bar{x}, \bar{p}\} = \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{x}}{\partial p} \frac{\partial \bar{p}}{\partial x} \quad (30)$$

$$= \cos^2 \theta - (-\sin^2 \theta) \quad (31)$$

$$= 1 \quad (32)$$

An infinitesimal rotation gives the transformation

$$\bar{x} = x - \varepsilon p \quad (33)$$

$$\bar{p} = p + \varepsilon x \quad (34)$$

To find the generator, we need to solve 13 and 14:

$$\frac{\partial g}{\partial p} = -p \quad (35)$$

$$\frac{\partial g}{\partial x} = -x \quad (36)$$

These can be integrated to give

$$g(x,p) = -\frac{1}{2}(p^2 + x^2) + C \quad (37)$$

where C is a constant of integration. Thus the quantity that is conserved is (apart from the minus sign, which we could eliminate by rotating through $-\theta$ instead of θ) is just the original Hamiltonian, or total energy.

PINGBACKS

Pingback: Hamilton's equations of motion under a regular canonical transformation