

INFINITESIMAL ROTATIONS IN CANONICAL AND NONCANONICAL TRANSFORMATIONS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercises 2.8.3 - 2.8.4.

Here are a couple of examples of transformations of variables and their consequences with regard to conservation laws.

First, we look at the 2-d harmonic oscillator where the Hamiltonian is

$$(0.1) \quad H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2} m \omega^2 (x^2 + y^2)$$

If we rotate the system so that both the coordinates and momenta get rotated, then

$$(0.2) \quad \bar{x} = x \cos \theta - y \sin \theta$$

$$(0.3) \quad \bar{y} = x \sin \theta + y \cos \theta$$

$$(0.4) \quad \bar{p}_x = p_x \cos \theta - p_y \sin \theta$$

$$(0.5) \quad \bar{p}_y = p_x \sin \theta + p_y \cos \theta$$

We can show by direct calculation that H is invariant under this transformation, and we can verify that this is a canonical transformation. Shankar shows in his equation 2.8.8 that the generator of this transformation is the angular momentum $\ell_z = xp_y - yp_x$.

However, if we rotate only the coordinates and not the momenta, we get the transformation:

$$(0.6) \quad \bar{x} = x \cos \theta - y \sin \theta$$

$$(0.7) \quad \bar{y} = x \sin \theta + y \cos \theta$$

$$(0.8) \quad \bar{p}_x = p_x$$

$$(0.9) \quad \bar{p}_y = p_y$$

Again, we can show by direct calculation that

$$(0.10) \quad \bar{x}^2 + \bar{y}^2 = x^2 + y^2$$

so H is also invariant under this transformation. However, this transformation is noncanonical, as we can see by calculating one of the Poisson brackets:

$$(0.11) \quad \{\bar{x}, \bar{p}_x\} = \sum_i \left(\frac{\partial \bar{x}}{\partial q_i} \frac{\partial \bar{p}_x}{\partial p_i} - \frac{\partial \bar{x}}{\partial p_i} \frac{\partial \bar{p}_x}{\partial q_i} \right)$$

$$(0.12) \quad = \cos \theta \neq 1$$

The other mixed brackets (with a coordinate and a momentum) are also not either 0 or 1 as would be required if the transformation were to be canonical.

In order for this transformation to give rise to a conservation law, we would need to find a generator g that satisfied, for an infinitesimal rotation ε :

$$(0.13) \quad \bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i$$

$$(0.14) \quad \bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i$$

For an infinitesimal rotation, the transformation 0.6 becomes

$$(0.15) \quad \bar{x} = x - \varepsilon y$$

$$(0.16) \quad \bar{y} = y + \varepsilon x$$

$$(0.17) \quad \bar{p}_x = p_x$$

$$(0.18) \quad \bar{p}_y = p_y$$

Therefore, the generator would have to satisfy

$$(0.19) \quad \frac{\partial g}{\partial p_x} = -y$$

$$(0.20) \quad \frac{\partial g}{\partial p_y} = x$$

$$(0.21) \quad \frac{\partial g}{\partial x} = 0$$

$$(0.22) \quad \frac{\partial g}{\partial y} = 0$$

The last two conditions state that g cannot depend on x or y , but integrating the first two conditions, we get

$$(0.23) \quad g = -yp_x + xp_y + f(x, y)$$

where f is a function that depends only on x and/or y . Thus there is no g that satisfies all four conditions, so there is no conservation law associated with a rotation of the coordinates only, even though the Hamiltonian is invariant under this transformation. Only canonical transformations that leave H invariant give rise to conservation laws.

As another example, suppose we have the one-dimensional system with

$$(0.24) \quad H = \frac{1}{2}(p^2 + x^2)$$

and perform a rotation in phase space, that is, in the $x - p$ plane:

$$(0.25) \quad \bar{x} = x \cos \theta - p \sin \theta$$

$$(0.26) \quad \bar{p} = x \sin \theta + p \cos \theta$$

The Hamiltonian is invariant:

$$(0.27) \quad \bar{p}^2 + \bar{x}^2 = x^2 \sin^2 \theta + 2xp \sin \theta \cos \theta + p^2 \cos^2 \theta +$$

$$(0.28) \quad x^2 \cos^2 \theta - 2xp \sin \theta \cos \theta + p^2 \sin^2 \theta$$

$$(0.29) \quad = x^2 + p^2$$

The transformation is canonical as we can verify by calculating the Poisson bracket

$$(0.30) \quad \{\bar{x}, \bar{p}\} = \frac{\partial \bar{x}}{\partial x} \frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{x}}{\partial p} \frac{\partial \bar{p}}{\partial x}$$

$$(0.31) \quad = \cos^2 \theta - (-\sin^2 \theta)$$

$$(0.32) \quad = 1$$

An infinitesimal rotation gives the transformation

$$(0.33) \quad \bar{x} = x - \varepsilon p$$

$$(0.34) \quad \bar{p} = p + \varepsilon x$$

To find the generator, we need to solve 0.13 and 0.14:

$$(0.35) \quad \frac{\partial g}{\partial p} = -p$$

$$(0.36) \quad \frac{\partial g}{\partial x} = -x$$

These can be integrated to give

$$(0.37) \quad g(x, p) = -\frac{1}{2}(p^2 + x^2) + C$$

where C is a constant of integration. Thus the quantity that is conserved is (apart from the minus sign, which we could eliminate by rotating through $-\theta$ instead of θ) is just the original Hamiltonian, or total energy.

PINGBACKS

Pingback: Hamilton's equations of motion under a regular canonical transformation