HAMILTON'S EQUATIONS OF MOTION UNDER A REGULAR CANONICAL TRANSFORMATION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercise 2.8.5.

If the Hamiltonian is invariant under a regular canonical transformation and we can find a generator g such that an infinitesimal version of this transformation is given by

$$\bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i \tag{1}$$

$$\bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i \tag{2}$$

then g is conserved.

If we are dealing with a *finite* regular canonical transformation where we go from $(q,p) \rightarrow (\bar{q},\bar{p})$, and the Hamiltonian is invariant under this transformation, then it turns out that if a trajectory $(q(t),p(t))$ satisfies Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{3}$$

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i \tag{4}$$

then the trajectory obtained by transforming every point in the original trajectory $(q(t),p(t))$ to the barred system $(\bar{q}(t),\bar{p}(t))$ is also a solution of Hamilton's equations in the sense that

$$\frac{\partial H}{\partial \bar{p}_i} = \dot{\bar{q}}_i \tag{5}$$

$$-\frac{\partial H}{\partial \bar{q}_i} = \dot{\bar{p}}_i \tag{6}$$

The proof of this is a bit subtle, but goes as follows. To begin, review the derivation of the conditions for a transformation to be canonical. This derivation applied to a passive transformation, in which the two sets of parameters $(q,p) \rightarrow (\bar{q},\bar{p})$ refer to the same point in phase space. The transformation we're considering here is an active transformation, in which $(q,p) \rightarrow (\bar{q},\bar{p})$ actually moves the point in phase space. The original derivation (for passive transformations) relied on the fact that the numerical value of the Hamiltonian is the same in both coordinate systems, since both (q,p) and $(\bar{q},\bar{p})$ refer to the same point in phase space. However, for our active transformation, we're assuming that the Hamiltonian is invariant under the transformation, that is $H(\bar{q},\bar{p}) = H(q,p)$, where (q,p) and $(\bar{q},\bar{p})$ now refer to *different* points in phase space. Since the assumption that the Hamiltonian satisfies $H(\bar{q},\bar{p}) = H(q,p)$ was all that we used in the original derivation, the same derivation works both for passive transformations (always) and for active transformations (if the Hamiltonian is invariant under the active transformation). We therefore end up with the equations

$$\dot{\overline{q}}_j = \sum_k \frac{\partial H}{\partial \overline{q}_k}\left\{\overline{q}_j, \overline{q}_k\right\} + \sum_k \frac{\partial H}{\partial \overline{p}_k}\left\{\overline{q}_j, \overline{p}_k\right\} \tag{7}$$

$$\dot{\overline{p}}_j = \sum_k \frac{\partial H}{\partial \overline{q}_k}\left\{\overline{p}_j, \overline{q}_k\right\} + \sum_k \frac{\partial H}{\partial \overline{p}_k}\left\{\overline{p}_j, \overline{p}_k\right\} \tag{8}$$

Since the transformation is specified to be canonical, the conditions on the Poisson brackets apply here:

$$\left\{\overline{q}_j, \overline{q}_k\right\} = \left\{\overline{p}_j, \overline{p}_k\right\} = 0 \tag{9}$$

$$\left\{\overline{q}_j, \overline{p}_k\right\} = \delta_{jk} \tag{10}$$

The result is that the transformed trajectory also satisfies Hamilton's equations 5 and 6.

We can now revisit the 2-d harmonic oscillator to show that a noncanonical transformation violates these results. The Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) \tag{11}$$

and we consider the transformation where we rotate the coordinates but not the momenta. The transformation is

$$\bar{x} = x\cos\theta - y\sin\theta \tag{12}$$

$$\bar{y} = x\sin\theta + y\cos\theta \tag{13}$$

$$\bar{p}_x = p_x \tag{14}$$

$$\bar{p}_y = p_y \tag{15}$$

As we've seen, this is a noncanonical transformation. To see what happens, we'll consider the initial conditions

$$x(0) = a \tag{16}$$
$$p_x(0) = b \tag{17}$$
$$y(0) = p_y(0) = 0 \tag{18}$$

The mass is started off at a point on the x axis with a momentum only in the x direction. In this case, the mass behaves like a one-dimensional harmonic oscillator, moving along the x axis only. To be precise, we can work out Hamilton's equations of motion:

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -m\omega^2 x \tag{19}$$
$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \tag{20}$$

The equations for y and p_y are the same, with x replaced by y everywhere. We can solve these ODEs in the usual way, by differentiating the first one and substituting the second one into the first to get

$$\ddot{p}_x = -m\omega^2\dot{x} = -\omega^2 p_x \tag{21}$$

This has the general solution

$$p_x(t) = A\cos\omega t + B\sin\omega t \tag{22}$$

We can do the same for x and get

$$x(t) = C\cos\omega t + D\sin\omega t \tag{23}$$

Applying the initial conditions, we get

$$p_x(0) = A = b \tag{24}$$
$$x(0) = C = a \tag{25}$$

Plugging these into the equations of motion 19 and 20 and solving for B and D we get the final solution

$$p_x(t) = b\cos\omega t - m\omega a\sin\omega t \tag{26}$$
$$x(t) = a\cos\omega t + \frac{b}{m\omega}\sin\omega t \tag{27}$$
$$y(t) = p_y(t) = 0 \tag{28}$$

Now suppose we start off with $x(0)=0$, $y(0)=a$, $p_x(0)=b$ and $p_y(0)=0$. That is, we have rotated the coordinates through $\frac{\pi}{2}$, but not the momenta. We now begin with the mass on the y axis, but moving in the x direction, so as time progresses, it will have components of momentum in both the x and y directions. Although it's fairly obvious that this motion will not be simply the motion in the first case rotated through $\frac{\pi}{2}$, let's go through the equations. By the same technique as above, we can solve the equations to get

$$p_x(t) = b\cos\omega t \tag{29}$$

$$p_y(t) = -m\omega a\sin\omega t \tag{30}$$

$$x(t) = \frac{b}{m\omega}\sin\omega t \tag{31}$$

$$y(t) = a\cos\omega t \tag{32}$$

If we look at the system at, say, $t=\frac{\pi}{2\omega}$, then $\cos\omega t=0$ and $\sin\omega t=1$. The mass that started off on the x axis will be at position $(x,y)=\left(\frac{b}{m\omega},0\right)$ and so will the mass that started off on the y axis. Since the two masses are in the same place, obviously one is not the rotated version of the other.

Another, probably easier, way to see this is that since the first mass moves only along the x axis, if the rotated version of the trajectory was also to be a solution, the rotated trajectory would have to lie entirely along the y axis, which is certainly not true for the mass that starts off on the y axis, but with a momentum $p_x \neq 0$.

In the general case, if the transformation is noncanonical, then the Poisson brackets in 7 and 8 don't satisfy the conditions 9 and 10, with the result that Hamilton's equations aren't satisfied in the $(\bar{q},\bar{p})$ coordinates. (There may be a deeper, physical interpretation that I've missed, but from a mathematical point of view, that's what goes wrong.)