RELATION BETWEEN ACTION AND ENERGY

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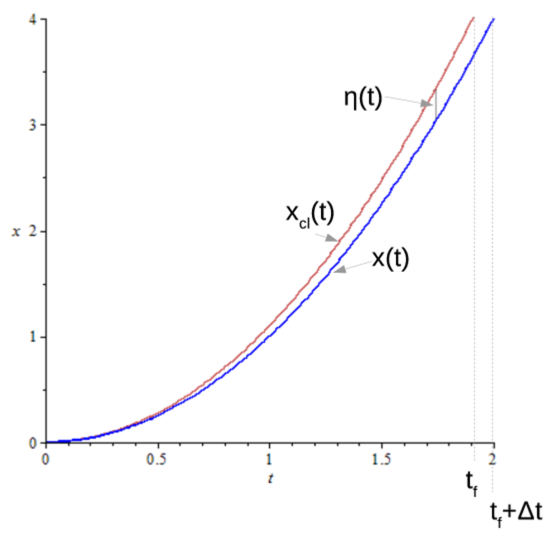
References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercises 2.8.6 - 2.8.7.

Here we'll examine an interesting relation between the action S and the total energy of a system, as given by the Hamiltonian H. Suppose a single particle moving in one dimension follows a classical path given by $x_{cl}(t)$, and moves from an initial position at time t_i of $x_{cl}(t_i) = x_i$ to a final position at time t_f of $x_{cl}(t_f) = x_f$. The action S_{cl} of this classical path is given by the integral of the Lagrangian

$$S_{cl} = \int_{t_i}^{t_f} L(x, \dot{x})\, dt \tag{1}$$

What can we say about the rate of change of the action with respect to the final time t_f? That is, we want to calculate $\partial S_{cl}/\partial t_f$, where all other parameters $t_i, x_i$and x_f are held constant. The situation can be illustrated as shown:



Since the only thing that is changing is t_f, the particle starts at the same initial time (which we've taken to be $t_i = 0$ in the diagram) and moves to the same location x_f, but at a different time (in the diagram, later time). This means that the particle must follow a different path, possibly over its entire trajectory. This path, which we'll call $x(t)$, is related to the original path $x_{cl}(t)$ by perturbing the original path by an amount $\eta(t)$:

$$x(t) = x_{cl}(t) + \eta(t) \tag{2}$$

In the diagram, the original path x_{cl} is shown in red and the perturbed path x in blue. The amount η is seen to be the vertical distance between these two curves at each time, and in the case of the paths shown in the diagram, $\eta(t) < 0$.

The difference in the action between the two paths is due to two contributions: first, there is the contribution due to the extra time, from t_f to $t_f + \Delta t$, that the particle takes to complete its path. Second, there is the difference in the two actions over the path from t_i to t_f. The first contribution is entirely new and, for an infinitesimal extra time Δt, it is given by

$$\delta S_1 = L\left(t_f\right)\Delta t \tag{3}$$

where $L\left(t_f\right)$ is the Lagrangian evaluated at time t_f. The other contribution can be obtained by varying the action over the path from $t_i = 0$ to t_f:

$$\delta S_2 = \int_0^{t_f} \delta L \, dt \tag{4}$$

Since L depends on x and $\dot{x}$, we have

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \tag{5}$$

For infinitesimally different trajectories, we can see from the diagram above that $\delta x = \eta(t)$ at each point on the curve, so $\delta \dot{x} = \dot{\eta}(t)$, so we get

$$\begin{aligned}
\delta S_2 &= \int_0^{t_f} \left[\frac{\partial L}{\partial x}\eta(t) + \frac{\partial L}{\partial \dot{x}}\dot{\eta}(t)\right] dt && (6)\\
&= \int_0^{t_f} \left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x}\right]\eta(t)\, dt + \int_0^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\eta(t)\right) dt && (7)\\
&= 0 + \left.\frac{\partial L}{\partial \dot{x}}\eta(t)\right|_{t_f} && (8)
\end{aligned}$$

In these equations, the derivatives of L are evaluated on the original curve x_{cl}. To verify the second line, use the product rule on the second integrand and cancel terms to get the first line. The second term in the last is evaluated at $t = t_f$ only since we're assuming that $\eta(0) = 0$.

The quantity in brackets in the first integral is zero, because of the Euler-Lagrange equations which are valid on the original curve x_{cl}:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x}=0 \tag{9}$$

Putting everything together, we get for the total variation in the action:

$$\begin{aligned}\delta S_{cl} &= \delta S_1+\delta S_2 &(10)\\ &= \left[\frac{\partial L}{\partial\dot{x}}\eta(t)+L\Delta t\right]_{t_f} &(11)\end{aligned}$$

Looking at the diagram above, the slope of the blue curve $x\left(t_f\right)$ at the time t_f is given by

$$\dot{x}\left(t_f\right)=\frac{\left|\eta\left(t_f\right)\right|}{\Delta t} \tag{12}$$

From the definition 2 of η we see that $\eta\left(t_f\right)<0$, so

$$\eta\left(t_f\right)=-\dot{x}\left(t_f\right)\Delta t \tag{13}$$

This gives the final equation for the variation of the action:

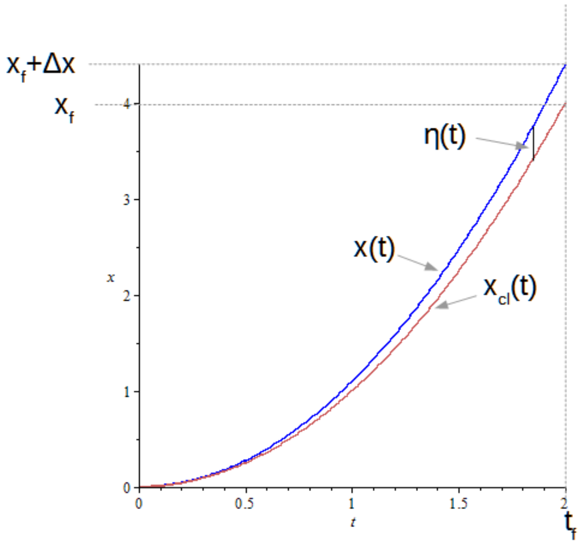
$$\begin{aligned}\delta S_{cl} &= \left[-\frac{\partial L}{\partial\dot{x}}\dot{x}+L\right]_{t_f}\Delta t &(14)\\ &= \left(-p\dot{x}+L\right)\Delta t &(15)\\ &= -H\Delta t &(16)\end{aligned}$$

where the second line follows from the definition of the canonical momentum $p=\partial L/\partial\dot{x}$.

The required derivative is

$$\boxed{\frac{\partial S_{cl}}{\partial t_f}=-H\left(t_f\right)} \tag{17}$$

Using a similar technique, we can work out $\partial S_{cl}/\partial x_f$. In this case, the situation is as shown in this diagram:



The two trajectories now take the same time, but in the modified trajectory, the particle moves a distance Δx further. Since both paths take the same time, there is no extra contribution $L\Delta t$. In this case $\eta(t) > 0$, since the new (blue) curve $x(t)$ is above the old (red) one $x_{cl}(t)$. The derivation is the same as above up to 8, and the total variation in the action is now

$$\delta S_{cl} = \left.\frac{\partial L}{\partial \dot{x}}\eta(t)\right|_{t_f} \tag{18}$$

At $t = t_f$, $\eta(t_f) = \Delta x$, so we get

$$\begin{aligned} \delta S_{cl} &= \left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} \Delta x && (19) \\ \frac{\partial S_{cl}}{\partial x_f} &= \left.\frac{\partial L}{\partial \dot{x}}\right|_{t_f} = p(t_f) && (20) \end{aligned}$$

Example. We can verify 17 for the case of the one-dimensional harmonic oscillator. The general solution for the position is given by

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t && (21) \\ \dot{x}(t) &= -A\omega\sin\omega t + B\omega\cos\omega t && (22) \end{aligned}$$

The total energy is given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 \tag{23}$$
$$= \frac{m}{2}\left((-A\omega\sin\omega t + B\omega\cos\omega t)^2 + \omega^2 (A\cos\omega t + B\sin\omega t)^2\right) \tag{24}$$
$$= \frac{m\omega^2}{2}\left(A^2 + B^2\right) \tag{25}$$

where we just multiplied out the second line, cancelled terms and used $\cos^2 x + \sin^2 x = 1$.

To get the action, we need the Lagrangian:

$$L = T - V \tag{26}$$
$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \tag{27}$$
$$= \frac{m}{2}\left((-A\omega\sin\omega t + B\omega\cos\omega t)^2 - \omega^2 (A\cos\omega t + B\sin\omega t)^2\right) \tag{28}$$
$$= \frac{m\omega^2}{2}\left[A^2\left(\sin^2\omega t - \cos^2\omega t\right) + B^2\left(\cos^2\omega t - \sin^2\omega t\right) - 4AB\sin\omega t\cos\omega t\right] \tag{29}$$
$$= \frac{m\omega^2}{2}\left(\left(B^2 - A^2\right)\cos 2\omega t - 2AB\sin 2\omega t\right) \tag{30}$$

The action for a trajectory from $t = 0$ to $t = T$ is then

$$S = \int_0^T L dt \tag{31}$$
$$= \frac{m\omega}{4}\left[\left(B^2 - A^2\right)\sin 2\omega t + 2AB\cos 2\omega t\right]_0^T \tag{32}$$
$$= \frac{m\omega}{4}\left[\left(B^2 - A^2\right)\sin 2\omega T + 2AB\left(\cos 2\omega T - 1\right)\right] \tag{33}$$
$$= \frac{m\omega}{2}\left[\left(B^2 - A^2\right)\sin\omega T\cos\omega T + AB\left(\cos^2\omega T - \sin^2\omega T - 1\right)\right] \tag{34}$$
$$= \frac{m\omega}{2}\left[\left(B^2 - A^2\right)\sin\omega T\cos\omega T - 2AB\sin^2\omega T\right] \tag{35}$$

To proceed further, we need to specify A and B, since these depend on the boundary conditions (that is, on where we require the mass to be at $t = 0$ and $t = T$). If we require $x(0) = x_1$ and $x(T) = x_2$, then

$$A = x_1 \tag{36}$$

$$x_1 \cos\omega T + B \sin\omega T = x_2 \tag{37}$$

$$B = \frac{x_2 - x_1 \cos\omega T}{\sin\omega T} \tag{38}$$

Plugging these into 25 gives the energy as

$$E = \frac{m\omega^2}{2}\left(x_1^2 + \left(\frac{x_2 - x_1\cos\omega T}{\sin\omega T}\right)^2\right) \tag{39}$$

$$= \frac{m\omega^2}{2\sin^2\omega T}\left(x_1^2 + x_2^2 - 2x_1x_2\cos\omega T\right) \tag{40}$$

Plugging A and B into 35, we get (using $c \equiv \cos\omega T$ and $s \equiv \sin\omega T$, so that $s^2 + c^2 = 1$):

$$S = \frac{m\omega}{2s}\left[(x_2 - x_1c)^2 c - x_1 s^2 c - 2x_1 s^2 (x_2 - x_1 c)\right] \tag{41}$$

$$= \frac{m\omega}{2s}\left[\left(x_2^2 - 2x_1x_2c + x_1^2c^2\right)c - x_1^2 s^2 c - 2x_1x_2s^2 + 2x_1s^2c\right] \tag{42}$$

$$= \frac{m\omega}{2s}\left[\left(x_1^2 + x_2^2\right)c - 2x_1x_2\right] \tag{43}$$

$$= \frac{m\omega}{2\sin\omega T}\left[\left(x_1^2 + x_2^2\right)\cos\omega T - 2x_1x_2\right] \tag{44}$$

Taking the derivative, we get

$$\frac{\partial S}{\partial T} = \frac{m\omega}{2s^2}\left[-\omega\left(x_1^2 + x_2^2\right)s^2 - \left(\left(x_1^2 + x_2^2\right)c - 2x_1x_2\right)\omega c\right] \tag{45}$$

$$= \frac{m\omega^2}{2s^2}\left[-\left(x_1^2 + x_2^2\right) + 2x_1x_2c\right] \tag{46}$$

$$= -\frac{m\omega^2}{2\sin^2\omega T}\left(x_1^2 + x_2^2 - 2x_1x_2\cos\omega T\right) \tag{47}$$

$$= -E \tag{48}$$

Thus the result is verified for the harmonic oscillator.

PINGBACKS

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