

RELATION BETWEEN ACTION AND ENERGY

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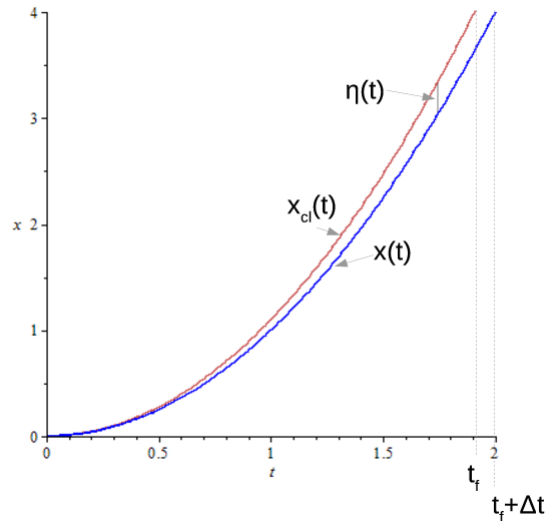
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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercises 2.8.6 - 2.8.7.

Here we'll examine an interesting relation between the action S and the total energy of a system, as given by the Hamiltonian H . Suppose a single particle moving in one dimension follows a classical path given by $x_{cl}(t)$, and moves from an initial position at time t_i of $x_{cl}(t_i) = x_i$ to a final position at time t_f of $x_{cl}(t_f) = x_f$. The action S_{cl} of this classical path is given by the integral of the Lagrangian

$$S_{cl} = \int_{t_i}^{t_f} L(x, \dot{x}) dt \quad (1)$$

What can we say about the rate of change of the action with respect to the final time t_f ? That is, we want to calculate $\partial S_{cl} / \partial t_f$, where all other parameters t_i, x_i and x_f are held constant. The situation can be illustrated as shown:



Since the only thing that is changing is t_f , the particle starts at the same initial time (which we've taken to be $t_i = 0$ in the diagram) and moves to the same location x_f , but at a different time (in the diagram, later time). This means that the particle must follow a different path, possibly over its entire

trajectory. This path, which we'll call $x(t)$, is related to the original path $x_{cl}(t)$ by perturbing the original path by an amount $\eta(t)$:

$$x(t) = x_{cl}(t) + \eta(t) \quad (2)$$

In the diagram, the original path x_{cl} is shown in red and the perturbed path x in blue. The amount η is seen to be the vertical distance between these two curves at each time, and in the case of the paths shown in the diagram, $\eta(t) < 0$.

The difference in the action between the two paths is due to two contributions: first, there is the contribution due to the extra time, from t_f to $t_f + \Delta t$, that the particle takes to complete its path. Second, there is the difference in the two actions over the path from t_i to t_f . The first contribution is entirely new and, for an infinitesimal extra time Δt , it is given by

$$\delta S_1 = L(t_f) \Delta t \quad (3)$$

where $L(t_f)$ is the Lagrangian evaluated at time t_f . The other contribution can be obtained by varying the action over the path from $t_i = 0$ to t_f :

$$\delta S_2 = \int_0^{t_f} \delta L dt \quad (4)$$

Since L depends on x and \dot{x} , we have

$$\delta L = \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \quad (5)$$

For infinitesimally different trajectories, we can see from the diagram above that $\delta x = \eta(t)$ at each point on the curve, so $\delta \dot{x} = \dot{\eta}(t)$, so we get

$$\delta S_2 = \int_0^{t_f} \left[\frac{\partial L}{\partial x} \eta(t) + \frac{\partial L}{\partial \dot{x}} \dot{\eta}(t) \right] dt \quad (6)$$

$$= \int_0^{t_f} \left[-\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} \right] \eta(t) dt + \int_0^{t_f} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \eta(t) \right) dt \quad (7)$$

$$= 0 + \left. \frac{\partial L}{\partial \dot{x}} \eta(t) \right|_{t_f} \quad (8)$$

In these equations, the derivatives of L are evaluated on the original curve x_{cl} . To verify the second line, use the product rule on the second integrand and cancel terms to get the first line. The second term in the last is evaluated at $t = t_f$ only since we're assuming that $\eta(0) = 0$.

The quantity in brackets in the first integral is zero, because of the Euler-Lagrange equations which are valid on the original curve x_{cl} :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad (9)$$

Putting everything together, we get for the total variation in the action:

$$\delta S_{cl} = \delta S_1 + \delta S_2 \quad (10)$$

$$= \left[\frac{\partial L}{\partial \dot{x}} \eta(t) + L \Delta t \right]_{t_f} \quad (11)$$

Looking at the diagram above, the slope of the blue curve $x(t_f)$ at the time t_f is given by

$$\dot{x}(t_f) = \frac{|\eta(t_f)|}{\Delta t} \quad (12)$$

From the definition 2 of η we see that $\eta(t_f) < 0$, so

$$\eta(t_f) = -\dot{x}(t_f) \Delta t \quad (13)$$

This gives the final equation for the variation of the action:

$$\delta S_{cl} = \left[-\frac{\partial L}{\partial \dot{x}} \dot{x} + L \right]_{t_f} \Delta t \quad (14)$$

$$= (-p\dot{x} + L) \Delta t \quad (15)$$

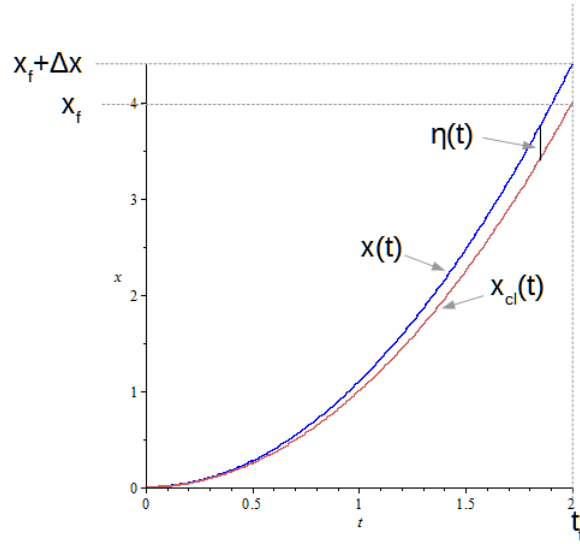
$$= -H \Delta t \quad (16)$$

where the second line follows from the definition of the canonical momentum $p = \partial L / \partial \dot{x}$.

The required derivative is

$$\boxed{\frac{\partial S_{cl}}{\partial t_f} = -H(t_f)} \quad (17)$$

Using a similar technique, we can work out $\partial S_{cl} / \partial x_f$. In this case, the situation is as shown in this diagram:



The two trajectories now take the same time, but in the modified trajectory, the particle moves a distance Δx further. Since both paths take the same time, there is no extra contribution $L\Delta t$. In this case $\eta(t) > 0$, since the new (blue) curve $x(t)$ is above the old (red) one $x_{cl}(t)$. The derivation is the same as above up to 8, and the total variation in the action is now

$$\delta S_{cl} = \left. \frac{\partial L}{\partial \dot{x}} \eta(t) \right|_{t_f} \quad (18)$$

At $t = t_f$, $\eta(t_f) = \Delta x$, so we get

$$\delta S_{cl} = \left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} \Delta x \quad (19)$$

$$\frac{\partial S_{cl}}{\partial x_f} = \left. \frac{\partial L}{\partial \dot{x}} \right|_{t_f} = p(t_f) \quad (20)$$

Example. We can verify 17 for the case of the one-dimensional harmonic oscillator. The general solution for the position is given by

$$x(t) = A \cos \omega t + B \sin \omega t \quad (21)$$

$$\dot{x}(t) = -A\omega \sin \omega t + B\omega \cos \omega t \quad (22)$$

The total energy is given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2 \quad (23)$$

$$= \frac{m}{2} \left((-A\omega \sin \omega t + B\omega \cos \omega t)^2 + \omega^2 (A \cos \omega t + B \sin \omega t)^2 \right) \quad (24)$$

$$= \frac{m\omega^2}{2} (A^2 + B^2) \quad (25)$$

where we just multiplied out the second line, cancelled terms and used $\cos^2 x + \sin^2 x = 1$.

To get the action, we need the Lagrangian:

$$L = T - V \quad (26)$$

$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 \quad (27)$$

$$= \frac{m}{2} \left((-A\omega \sin \omega t + B\omega \cos \omega t)^2 - \omega^2 (A \cos \omega t + B \sin \omega t)^2 \right) \quad (28)$$

$$= \frac{m\omega^2}{2} [A^2 (\sin^2 \omega t - \cos^2 \omega t) + B^2 (\cos^2 \omega t - \sin^2 \omega t) - 4AB \sin \omega t \cos \omega t] \quad (29)$$

$$= \frac{m\omega^2}{2} ((B^2 - A^2) \cos 2\omega t - 2AB \sin 2\omega t) \quad (30)$$

The action for a trajectory from $t = 0$ to $t = T$ is then

$$S = \int_0^T L dt \quad (31)$$

$$= \frac{m\omega}{4} [(B^2 - A^2) \sin 2\omega t + 2AB \cos 2\omega t]_0^T \quad (32)$$

$$= \frac{m\omega}{4} [(B^2 - A^2) \sin 2\omega T + 2AB (\cos 2\omega T - 1)] \quad (33)$$

$$= \frac{m\omega}{2} [(B^2 - A^2) \sin \omega T \cos \omega T + AB (\cos^2 \omega T - \sin^2 \omega T - 1)] \quad (34)$$

$$= \frac{m\omega}{2} [(B^2 - A^2) \sin \omega T \cos \omega T - 2AB \sin^2 \omega T] \quad (35)$$

To proceed further, we need to specify A and B , since these depend on the boundary conditions (that is, on where we require the mass to be at $t = 0$ and $t = T$). If we require $x(0) = x_1$ and $x(T) = x_2$, then

$$A = x_1 \quad (36)$$

$$x_1 \cos \omega T + B \sin \omega T = x_2 \quad (37)$$

$$B = \frac{x_2 - x_1 \cos \omega T}{\sin \omega T} \quad (38)$$

Plugging these into 25 gives the energy as

$$E = \frac{m\omega^2}{2} \left(x_1^2 + \left(\frac{x_2 - x_1 \cos \omega T}{\sin \omega T} \right)^2 \right) \quad (39)$$

$$= \frac{m\omega^2}{2 \sin^2 \omega T} (x_1^2 + x_2^2 - 2x_1 x_2 \cos \omega T) \quad (40)$$

Plugging A and B into 35, we get (using $c \equiv \cos \omega T$ and $s \equiv \sin \omega T$, so that $s^2 + c^2 = 1$):

$$S = \frac{m\omega}{2s} \left[(x_2 - x_1 c)^2 c - x_1 s^2 c - 2x_1 s^2 (x_2 - x_1 c) \right] \quad (41)$$

$$= \frac{m\omega}{2s} \left[(x_2^2 - 2x_1 x_2 c + x_1^2 c^2) c - x_1^2 s^2 c - 2x_1 x_2 s^2 + 2x_1 s^2 c \right] \quad (42)$$

$$= \frac{m\omega}{2s} \left[(x_1^2 + x_2^2) c - 2x_1 x_2 \right] \quad (43)$$

$$= \frac{m\omega}{2 \sin \omega T} \left[(x_1^2 + x_2^2) \cos \omega T - 2x_1 x_2 \right] \quad (44)$$

Taking the derivative, we get

$$\frac{\partial S}{\partial T} = \frac{m\omega}{2s^2} \left[-\omega (x_1^2 + x_2^2) s^2 - ((x_1^2 + x_2^2) c - 2x_1 x_2) \omega c \right] \quad (45)$$

$$= \frac{m\omega^2}{2s^2} \left[-(x_1^2 + x_2^2) + 2x_1 x_2 c \right] \quad (46)$$

$$= -\frac{m\omega^2}{2 \sin^2 \omega T} (x_1^2 + x_2^2 - 2x_1 x_2 \cos \omega T) \quad (47)$$

$$= -E \quad (48)$$

Thus the result is verified for the harmonic oscillator.

PINGBACKS

Pingback: Path integrals for special potentials; use of classical action