# POSTULATES OF QUANTUM MECHANICS: STATES AND MEASUREMENTS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Sections 4.1 - 4.2; Exercise 4.2.1.

Although we've covered the basics of nonrelativistic quantum mechanics before, the approach taken by Shankar in his Chapter 4 provides a new way of looking at it, so it's worth a summary.

Quantum mechanics is based on four postulates, the first three of which describe the quantum state at a fixed instant in time, and the fourth which describes its time evolution via the Schrödinger equation. We'll summarize the first three postulates here, and compare each with its classical analogue.

First, in classical mechanics, the path of a particle is, in the Hamiltonian formalism, described by specifying its position $x(t)$ and momentum $p(t)$ as functions of time. Both the position and momentum are specified precisely at all times. In quantum mechanics, the state of a particle is specified by a vector (ket) $|\psi(t)\rangle$ in a Hilbert space.

Second, in classical mechanics, any dynamical variable $\omega$ is a function of the two phase-space coordinates $x$ and $p$: $\omega=\omega(x,p)$. In quantum mechanics, the spatial coordinate $x$ is replaced by the Hermitian operator $X$ and the momentum $p$ is replaced by the differential operator $P=\hbar K$ which we discussed earlier. The matrix elements of $X$ and $P$ in position space are

$$\left\langle x\left|X\right|x'\right\rangle = x\delta\left(x-x'\right) \tag{1}$$

$$\left\langle x\left|P\right|x'\right\rangle = -i\hbar\delta'\left(x-x'\right) \tag{2}$$

The classical dynamical variable $\omega(x,p)$ becomes a Hermitian operator $\Omega(X,P)$, where $x$ and $p$ in $\omega(x,p)$ are replaced by their corresponding operators $X$ and $P$.

The third postulate states how measurements work in quantum mechanics. In classical mechanics, it is assumed that (in principle) any dynamical variable $\omega$ may be measured with arbitrary precision without changing the state of the particle. In quantum mechanics, if we wish to measure the value of a variable represented by the operator $\Omega$, we must determine the eigenvalues $\omega_i$ and corresponding eigenvectors $|\omega_i\rangle$ of $\Omega$, then express the state $|\psi\rangle$ as a linear combination of the $|\omega_i\rangle$. Then the best we can do is to state that the particular eigenvalue $\omega_i$ will be measured with probability $|\langle\omega_i|\psi\rangle|^2$. After the measurement, the state $|\psi\rangle$ 'collapses' to become the state $|\omega_i\rangle$. The only possible outcomes of a measurement of $\Omega$ are its eigenvalues; no intermediate values are possible.

To illustrate these postulates, suppose we have the following three operators on a complex 3-d Hilbert space (essentially these are the spin-1 operators without the $\hbar$)

$$L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3}$$

$$L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{4}$$

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{5}$$

Since $L_z$ is diagonal, its eigenvalues can be read off from the diagonal elements as $0, \pm 1$, so these are the possible values of $L_z$ that could be obtained in a measurement. Also because $L_z$ is diagonal, its eigenvectors are

$$|L_z = +1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$|L_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

$$|L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

Suppose we start with the state $|L_z = +1\rangle$ in which $L_z = +1$, and we want to measure $L_x$ in this state. To find the expectation values $\langle L_x\rangle$ and $\left\langle L_x^2\right\rangle$ in this state, we calculate

$$\langle L_x\rangle = \langle L_z = +1\,|L_x|\,L_z = +1\rangle \tag{9}$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

$$= 0 \tag{12}$$

To get $\left\langle L_x^2 \right\rangle$ we first find the operator

$$L_x^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{13}$$

Now we have

$$\left\langle L_x^2 \right\rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{15}$$

$$= \frac{1}{2} \tag{16}$$

The uncertainty, or variance, is

$$\Delta L_x = \sqrt{\langle L_x^2\rangle - \langle L_x\rangle^2} = \frac{1}{\sqrt{2}} \tag{17}$$

To find the possible values of $L_x$ and their probabilities, we need to find the eigenvalues and eigenvectors of $L_x$, which we can do in the $L_z$ basis, since this basis is given by the three vectors in 6. The eigenvalues are found in the usual way from the determinant:

$$\begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2-\frac{1}{2}\right)-\frac{1}{\sqrt{2}}\left(\frac{-\lambda}{\sqrt{2}}\right) \tag{18}$$

$$= -\lambda^3+\lambda = 0 \tag{19}$$

$$\lambda = 0, \pm 1 \tag{20}$$

The eigenvectors can be found in the usual way, by solving

$$(L_x-\lambda I)\,|L_x=\lambda\rangle = 0 \tag{21}$$

where the ket takes on the three possible values of $\lambda$ successively. We let

$$|L_x=\lambda\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{22}$$

For $\lambda=+1$ we have

$$-a+\frac{b}{\sqrt{2}} = 0 \tag{23}$$

$$\frac{1}{\sqrt{2}}\left(a-\sqrt{2}b+c\right) = 0 \tag{24}$$

$$\frac{b}{\sqrt{2}}-c = 0 \tag{25}$$

Only two of these three equations are independent, so we can set $a=1$ and solve for $b$ and $c$ to get

$$a = 1 \tag{26}$$

$$b = \sqrt{2} \tag{27}$$

$$c = 1 \tag{28}$$

Normalizing the eigenvector gives

$$|L_x=+1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \tag{29}$$

The other two eigenvectors can be found the same way, with the result

$$|L_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{30}$$

$$|L_x = -1\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \tag{31}$$

Note that these eigenvectors are orthonormal.

Now that we have the eigenvectors of $L_x$ we can answer the following question. If we start with the state $|L_z = -1\rangle$ and measure $L_x$, what are the possible outcomes and the probability of each?

First, we need to express $|L_z = -1\rangle$ in terms of the eigenvectors of $L_x$ which we can do by solving three simultaneous linear equations, and we find

$$|L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\left(|L_x = +1\rangle + |L_x = -1\rangle\right) - \frac{1}{\sqrt{2}}|L_x = 0\rangle \tag{32}$$

(You can verify this by direct substitution.) Thus all 3 possible values of $L_x$ can result from a measurement, and the probability of each is

$$P(L_x=+1) = |\langle L_x=+1\,|L_z=-1\,\rangle|^2 \tag{33}$$

$$= \left(\frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)^2 \tag{34}$$

$$= \frac{1}{4} \tag{35}$$

$$P(L_x=0) = |\langle L_x=0\,|L_z=-1\,\rangle|^2 \tag{36}$$

$$= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)^2 \tag{37}$$

$$= \frac{1}{2} \tag{38}$$

$$P(L_x=-1) = |\langle L_x=-1\,|L_z=-1\,\rangle|^2 \tag{39}$$

$$= \left(\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)^2 \tag{40}$$

$$= \frac{1}{4} \tag{41}$$

Now suppose we start with the state, written in the $L_z$ basis:

$$|\psi\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{42}$$

We take a measurement of $L_z^2$ and obtain $+1$. The operator $L_z^2$ is given by squaring 5:

$$L_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

This has a degenerate eigenvalue $\lambda=+1$, so the most we can say about the state $|\psi\rangle$ after the measurement is that it is projected onto the subspace $a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. That is, the state after the measurement is given by

$$|\psi\rangle_{after} = \mathbb{P}_{L_z=\pm1}\,|\psi\rangle_{before} \tag{44}$$

$$= \left[|L_z=+1\rangle\langle L_z=+1| + |L_z=-1\rangle\langle L_z=-1|\right]\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{45}$$

$$= \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{46}$$

$$= \begin{bmatrix}\frac{1}{2}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{47}$$

We can normalize this state to get

$$|\psi\rangle_{after} = \frac{2}{\sqrt{3}}\begin{bmatrix}\frac{1}{2}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{48}$$

Thus if we measure $L_z$ immediately after the measurement of $L_z^2$ above, we get $L_z=+1$ with probability $\frac{1}{3}$ and $L_z=-1$ with probability $\frac{2}{3}$.

Finally, suppose we have a state $|\psi\rangle$ with the probabilities of measurements of $L_z$ given as $P(L_z=1)=\frac{1}{4}$, $P(L_z=0)=\frac{1}{2}$ and $P(L_z=-1)=\frac{1}{4}$. Since these probabilities are given by $|\langle L_z=\lambda\,|\psi\rangle|^2$ for each of the three possible values of $\lambda$, and the vectors $|L_z=\lambda\rangle$ are orthonormal, the most general form for $|\psi\rangle$ is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z=1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z=0\rangle + \frac{e^{i\delta_3}}{2}|L_z=-1\rangle \tag{49}$$

where the $\delta_i$ are real numbers. For example

$$|\langle L_z=1\,|\psi\rangle|^2 = \left|\frac{e^{i\delta_1}}{2}\right|^2 = \frac{1}{4} \tag{50}$$

While the presence of a phase factor in a solitary state doesn't affect the physics of that state, if we have a sum of states, each with its own (different) phase factor, we can't ignore these phase factors. For example, if we measure $L_x$ in this state and want the probability that $L_x=0$, we have, using 30

(51)

$$P\left(L_x=0\right)=\left|\left\langle L_x=0\,|\,\psi\right\rangle\right|^2$$

(52)

$$=\left|\frac{1}{\sqrt{2}}\left[\begin{array}{ccc}1 & 0 & -1\end{array}\right]\left(\frac{e^{i\delta_1}}{2}\left|L_z=1\right\rangle+\frac{e^{i\delta_2}}{\sqrt{2}}\left|L_z=0\right\rangle+\frac{e^{i\delta_3}}{2}\left|L_z=-1\right\rangle\right)\right|^2$$

(53)

$$=\left|\frac{1}{\sqrt{2}}\left[\begin{array}{ccc}1 & 0 & -1\end{array}\right]\left(\frac{e^{i\delta_1}}{2}\left[\begin{array}{c}1\\0\\0\end{array}\right]+\frac{e^{i\delta_2}}{\sqrt{2}}\left[\begin{array}{c}0\\1\\0\end{array}\right]+\frac{e^{i\delta_3}}{2}\left[\begin{array}{c}0\\0\\1\end{array}\right]\right)\right|^2$$

(54)

$$=\frac{1}{8}\left|e^{i\delta_1}-e^{i\delta_3}\right|^2$$

(55)

$$=\frac{1}{8}\left|1-e^{i(\delta_3-\delta_1)}\right|^2$$

The last line will have a different result for different values of the phase factors $\delta_1$ and $\delta_3$, so they can't be ignored.

## PINGBACKS

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