POSTULATES OF QUANTUM MECHANICS: MOMENTUM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Sections 4.1 - 4.2; Exercises 4.2.2 - 4.2.3.

One of the postulates of quantum mechanics is that the momentum operator P in position space is given by

$$\left\langle x\left|P\right|x^{\prime}\right\rangle =-i\hbar\delta^{\prime}\left(x-x^{\prime}\right) \tag{1}$$

By using the properties of the derivative of the delta function, we can find the eigenfunctions of P. We have

$$\begin{aligned}
\left\langle x\left|P\right|\psi\right\rangle &= \int\left\langle x\left|P\right|x^{\prime}\right\rangle \left\langle x^{\prime}\left|\psi\right.\right\rangle dx^{\prime} && (2)\\
&= -i\hbar\int\delta^{\prime}\left(x-x^{\prime}\right)\left\langle x^{\prime}\left|\psi\right.\right\rangle dx^{\prime} && (3)\\
&= -i\hbar\frac{d}{dx}\left\langle x\left|\psi\right.\right\rangle && (4)\\
&= -i\hbar\frac{d\psi\left(x\right)}{dx} && (5)
\end{aligned}$$

The eigenvector of P is $\left|p\right\rangle$ and has the property that

$$P\left|p\right\rangle =p\left|p\right\rangle \tag{6}$$

If we project this onto position space and use 5 we get

$$\begin{aligned}
\left\langle x\left|P\right|\psi\right\rangle &= p\left\langle x\left|p\right.\right\rangle && (7)\\
-i\hbar\frac{d\psi_{p}\left(x\right)}{dx} &= p\psi_{p}\left(x\right) && (8)
\end{aligned}$$

where

$$\psi_{p}\left(x\right)\equiv\left\langle x\left|p\right.\right\rangle \tag{9}$$

Solving this differential equation and normalizing so that $\left\langle p^{\prime}\left|p\right.\right\rangle =\delta\left(p-p^{\prime}\right)$ we get

$$\psi_{p}\left(x\right)=\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} \tag{10}$$

For an arbitrary wave function $|\psi\rangle$, if we know its position-space form, we can find its momentum-space version as follows:

$$\langle p|\psi\rangle = \int \langle p|x\rangle \langle x|\psi\rangle\, dx \tag{11}$$

$$= \int \psi_p^*(x) \langle x|\psi\rangle\, dx \tag{12}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x)\, dx \tag{13}$$

This has an interesting consequence if the position-space function $\psi(x)$ is real. The probability density for finding a particle in a state with momentum p is $|\langle p|\psi\rangle|^2$, which we can write as

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \tag{14}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{ip(x-x')/\hbar} \psi(x)\, \psi(x')\, dx\, dx' \tag{15}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x'-x)/\hbar} \psi(x)\, \psi(x')\, dx\, dx' \tag{16}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x-x')/\hbar} \psi(x')\, \psi(x)\, dx\, dx' \tag{17}$$

$$= |\langle -p|\psi\rangle|^2 \tag{18}$$

In the fourth line, since x and x' are dummy integration variables, both of which are integrated over the same range, we can simply swap them without changing anything. Note that the derivation relies on $\psi(x)$ being real, since if it were complex we would have

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \tag{19}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{ip(x-x')/\hbar} \psi(x)\, \psi^*(x')\, dx\, dx' \tag{20}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x'-x)/\hbar} \psi(x)\, \psi^*(x')\, dx\, dx' \tag{21}$$

$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x-x')/\hbar} \psi(x')\, \psi^*(x)\, dx\, dx' \tag{22}$$

$$\neq |\langle -p|\psi\rangle|^2 \tag{23}$$

since

$$|\langle -p|\psi\rangle|^2 = \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi(x)\,\psi^*\left(x'\right)dx\,dx' \tag{24}$$

That is, for $|\langle -p|\psi\rangle|^2$ the position x' that is the argument of the $\psi^*(x')$ factor appears as the positive term ipx' in the exponential, but in 22 the argument of the complex conjugate wave function is x, which appears as the negative term $-ipx$ in the exponential.

Thus for any real wave function, the probability of the particle having momentum $+p$ is equal to the probability of it having $-p$, so for such wave functions, the mean momentum is always $\langle P\rangle = 0$.

As another example, suppose we have a wave function $\psi(x)$ with a mean momentum $\bar{p}$, so that

$$\langle\psi|P|\psi\rangle = \bar{p} \tag{25}$$

If we now multiply ψ by $e^{ip_0x/\hbar}$ where p_0 is a constant momentum, we can calculate the new mean momentum using 5:

$$\langle P\rangle = \left\langle e^{ip_0x/\hbar}\psi\,|P|\,e^{ip_0x/\hbar}\psi\right\rangle \tag{26}$$

$$= -i\hbar\int e^{-ip_0x/\hbar}\psi^*(x)\frac{d}{dx}\left(e^{ip_0x/\hbar}\psi(x)\right)dx \tag{27}$$

$$= -i\hbar\int e^{-ip_0x/\hbar}\psi^*\left[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x)+e^{ip_0x/\hbar}\frac{d}{dx}\psi(x)\right]dx \tag{28}$$

$$= \int p_0\psi^*\psi dx - i\hbar\int\psi^*(x)\frac{d}{dx}\psi(x)\,dx \tag{29}$$

$$= p_0+\bar{p} \tag{30}$$

The first integral in the fourth line uses the fact that p_0 is constant and ψ is normalized so that

$$\int\psi^*\psi dx = 1 \tag{31}$$

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