FREE PARTICLE REVISITED: SOLUTION IN TERMS OF A PROPAGATOR

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.1, Exercise 5.1.1.

Having reviewed the background mathematics and postulates of quantum mechanics as set out by Shankar, we can now revisit some of the classic problems in non-relativistic quantum mechanics using Shankar's approach, as opposed to that of Griffiths that we've already studied.

The first problem we'll look it is that of the free particle. Following the fourth postulate, we write down the classical Hamiltonian for a free particle, which is

$$H=\frac{p^{2}}{2m} \tag{1}$$

where p is the momentum (we're working in one dimension) and m is the mass. To get the quantum version, we replace p by the momentum operator P and insert the result into the Schrödinger equation:

$$\begin{aligned} i\hbar\left|\dot{\psi}\right\rangle &= H\left|\psi\right\rangle \quad (2)\\ &= \frac{P^{2}}{2m}\left|\psi\right\rangle \quad (3)\end{aligned}$$

Since H is time-independent, the solution can be written using a propagator:

$$\left|\psi\left(t\right)\right\rangle =U\left(t\right)\left|\psi\left(0\right)\right\rangle \tag{4}$$

To find U, we need to solve the eigenvalue equation for the stationary states

$$\frac{P^{2}}{2m}\left|E\right\rangle =E\left|E\right\rangle \tag{5}$$

where E is an eigenvalue representing the allowable energies. Since the Hamiltonian is $P^{2}/2m$, and an eigenstate of P with eigenvalue p is also an eigenstate of P^2 with eigenvalue p^2, we can write this equation in terms of the momentum eigenstates $|p\rangle$:

$$\frac{P^2}{2m}|p\rangle = E|p\rangle \tag{6}$$

Using $P^2|p\rangle = p^2|p\rangle$ this gives

$$\left(\frac{p^2}{2m} - E\right)|p\rangle = 0 \tag{7}$$

Assuming that $|p\rangle$ is not a null vector gives the relation between momentum and energy:

$$p = \pm\sqrt{2mE} \tag{8}$$

Thus each allowable energy E has two possible momenta. Once we specify the momentum, we also specify the energy and since each energy state is two-fold degenerate, we can eliminate the ambiguity by specifying only the momentum. Therefore the propagator can be written as

$$U(t) = \int_{-\infty}^{\infty} e^{-ip^2t/2m\hbar}|p\rangle\langle p|\,dp \tag{9}$$

We can convert this to an integral over the energy by using 8 to change variables, and by splittling the integral into two parts. For $p > 0$ we have

$$dp = \sqrt{\frac{m}{2E}}dE \tag{10}$$

and for $p < 0$ we have

$$dp = -\sqrt{\frac{m}{2E}}dE \tag{11}$$

Therefore, we get

(12)
$$U(t)=\int_0^\infty e^{-iEt/\hbar}\left|E,+\right\rangle\left\langle E,+\right|\sqrt{\frac{m}{2E}}dE+\int_\infty^0 e^{-iEt/\hbar}\left|E,-\right\rangle\left\langle E,-\right|\left(-\sqrt{\frac{m}{2E}}\right)dE$$
(13)
$$=\int_0^\infty e^{-iEt/\hbar}\left|E,+\right\rangle\left\langle E,+\right|\sqrt{\frac{m}{2E}}dE+\int_0^\infty e^{-iEt/\hbar}\left|E,-\right\rangle\left\langle E,-\right|\sqrt{\frac{m}{2E}}dE$$
(14)
$$=\sum_{\alpha=\pm}\int_0^\infty\frac{m}{\sqrt{2mE}}e^{-iEt/\hbar}\left|E,\alpha\right\rangle\left\langle E,\alpha\right|dE$$

Here, $\left|E,+\right\rangle$ is the state with energy E and momentum $p=+\sqrt{2mE}$ and similarly for $\left|E,-\right\rangle$. In the first line, the first integral is for $p>0$ and corresponds to the $\int_0^\infty$ part of 9. The second integral is for $p<0$ and corresponds to the $\int_{-\infty}^0$ part of 9, which is why the limits on the second integral have ∞ at the bottom and 0 at the top. Reversing the order of integration cancels out the minus sign in $-\sqrt{\frac{m}{2E}}$, which allows us to add the two integrals together to get the final answer.

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