

PROBABILITY CURRENT WITH COMPLEX POTENTIAL

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.3, Exercise 5.3.1.

Shakar's derivation of the probability current in 3-d is similar to the one we reviewed earlier, so we don't need to repeat it here. We can, however, look at a slight variant where the potential has a constant imaginary part, so that

$$V(\mathbf{r}) = V_r(\mathbf{r}) - iV_i \quad (1)$$

where $V_r(\mathbf{r})$ is a real function of position and V_i is a real constant. A Hamiltonian containing such a complex potential is not Hermitian.

To see what effect this has on the total probability of finding a particle in all space, we can repeat the derivation of the probability current. From the Schrödinger equation and its complex conjugate, we have

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_r \psi - iV_i \psi \quad (2)$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V_r \psi^* + iV_i \psi^* \quad (3)$$

Multiply the first equation by ψ^* and the second by ψ and subtract to get

$$i\hbar \frac{\partial}{\partial t} (\psi \psi^*) = -\frac{\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) - 2iV_i \psi \psi^* \quad (4)$$

As in the case with a real potential, the first term on the RHS can be written as the divergence of a vector:

$$\mathbf{J} = \frac{\hbar}{2mi} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \quad (5)$$

$$\nabla \cdot \mathbf{J} = \frac{\hbar}{2mi} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) \quad (6)$$

$$\frac{\partial}{\partial t} (\psi \psi^*) = -\nabla \cdot \mathbf{J} - \frac{2V_i}{\hbar} \psi \psi^* \quad (7)$$

If we define the total probability of finding the particle anywhere in space as

$$P \equiv \int \psi^* \psi d^3 \mathbf{r} \quad (8)$$

then we can integrate 4 over all space and use Gauss's theorem to convert the volume integral of a divergence into a surface integral:

$$\frac{\partial}{\partial t} \left(\int \psi \psi^* d^3 \mathbf{r} \right) = - \int \nabla \cdot \mathbf{J} d^3 \mathbf{r} - \frac{2V_i}{\hbar} \int \psi \psi^* d^3 \mathbf{r} \quad (9)$$

$$\frac{\partial P}{\partial t} = - \int_S \mathbf{J} \cdot d\mathbf{a} - \frac{2V_i}{\hbar} P \quad (10)$$

We make the usual assumption that the probability current \mathbf{J} tends to zero at infinity fast enough for the first integral on the RHS to be zero, and we get

$$\frac{\partial P}{\partial t} = - \frac{2V_i}{\hbar} P \quad (11)$$

This has the solution

$$P(t) = P(0) e^{-2V_i t / \hbar} \quad (12)$$

That is, the probability of the particle existing decays exponentially. Although Shankar says that such a potential can be used to model a system where particles are absorbed, it's not clear how realistic it is since the Hamiltonian isn't hermitian, so technically the energies in such a system are not observables.