PROBABILITY CURRENT WITH COMPLEX POTENTIAL

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.3, Exercise 5.3.1.

Shakar's derivation of the probability current in 3-d is similar to the one we reviewed earlier, so we don't need to repeat it here. We can, however, look at a slight variant where the potential has a constant imaginary part, so that

$$V(\mathbf{r}) = V_r(\mathbf{r}) - iV_i \tag{1}$$

where $V_r(\mathbf{r})$ is a real function of position and V_i is a real constant. A Hamiltonian containing such a complex potential is not Hermitian.

To see what effect this has on the total probability of finding a particle in all space, we can repeat the derivation of the probability current. From the Schrödinger equation and its complex conjugate, we have

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_r\psi - iV_i\psi \tag{2}$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V_r\psi^* + iV_i\psi^* \tag{3}$$

Multiply the first equation by ψ^* and the second by ψ and subtract to get

$$i\hbar\frac{\partial}{\partial t}(\psi\psi^*) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) - 2iV_i\psi\psi^* \tag{4}$$

As in the case with a real potential, the first term on the RHS can be written as the divergence of a vector:

$$\mathbf{J} = \frac{\hbar}{2mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \tag{5}$$

$$\nabla\cdot\mathbf{J} = \frac{\hbar}{2mi}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \tag{6}$$

$$\frac{\partial}{\partial t}(\psi\psi^*) = -\nabla\cdot\mathbf{J} - \frac{2V_i}{\hbar}\psi\psi^* \tag{7}$$

If we define the total probability of finding the particle anywhere in space as

$$P\equiv\int\psi^{*}\psi d^{3}\mathbf{r} \tag{8}$$

then we can integrate 4 over all space and use Gauss's theorem to convert the volume integral of a divergence into a surface integral:

$$\frac{\partial}{\partial t}\left(\int\psi\psi^{*}d^{3}\mathbf{r}\right)=-\int\nabla\cdot\mathbf{J}d^{3}\mathbf{r}-\frac{2V_{i}}{\hbar}\int\psi\psi^{*}d^{3}\mathbf{r} \tag{9}$$

$$\frac{\partial P}{\partial t}=-\int_{S}\mathbf{J}\cdot d\mathbf{a}-\frac{2V_{i}}{\hbar}P \tag{10}$$

We make the usual assumption that the probability current $\mathbf{J}$ tends to zero at infinity fast enough for the first integral on the RHS to be zero, and we get

$$\frac{\partial P}{\partial t}=-\frac{2V_{i}}{\hbar}P \tag{11}$$

This has the solution

$$P(t)=P(0)\,e^{-2V_{i}t/\hbar} \tag{12}$$

That is, the probability of the particle existing decays exponentially. Although Shankar says that such a potential can be used to model a system where particles are absorbed, it's not clear how realistic it is since the Hamiltonian isn't hermitian, so technically the energies in such a system are not observables.