PROBABILITY CURRENT: A FEW EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.3, Exercises 5.3.2 - 5.3.4.

Here are a few examples of probability current.

Example 1. Suppose the wave function has the form

$$\psi(\mathbf{r},t)=c\tilde{\psi}(\mathbf{r},t) \tag{0.1}$$

where c is a complex constant and $\tilde{\psi}(\mathbf{r},t)$ is a real function of position and time. Then the probability current is

$$\mathbf{j}=\frac{\hbar}{2mi}\left(\psi^{*}\nabla\psi-\psi\nabla\psi^{*}\right) \tag{0.2}$$

$$=\frac{\hbar}{2mi}\left(cc^{*}\left(\tilde{\psi}\nabla\tilde{\psi}\right)-\tilde{\psi}\nabla\tilde{\psi}\right) \tag{0.3}$$

$$=0 \tag{0.4}$$

In particular, if ψ itself is real, the probability current is always zero, so all the stationary states of systems like the harmonic oscillator and hydrogen atom that we've studied show no flow of probability, which is what we'd expect since they are, after all, stationary states.

Example 2. Now the wave function is

$$\psi_{\mathbf{p}}\;=\;\frac{1}{(2\pi\hbar)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{0.5}$$

where the momentum $\mathbf{p}$ is constant. In this case we have

$$\nabla\psi_{\mathbf{p}}=\frac{i}{(2\pi\hbar)^{3/2}\hbar}e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\mathbf{p} \tag{0.6}$$

$$\nabla\psi_{\mathbf{p}}^{*}=\frac{-i}{(2\pi\hbar)^{3/2}\hbar}e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\mathbf{p} \tag{0.7}$$

$$\psi_{\mathbf{p}}^{*}=\frac{1}{(2\pi\hbar)^{3/2}}e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{0.8}$$

This gives a probability current of

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\psi_{\mathbf{p}}^{*}\nabla\psi_{\mathbf{p}} - \psi_{\mathbf{p}}\nabla\psi_{\mathbf{p}}^{*}\right) \tag{0.9}$$

$$= \frac{1}{(2\pi\hbar)^{3}\,2m}(\mathbf{p}+\mathbf{p}) \tag{0.10}$$

$$= \frac{1}{(2\pi\hbar)^{3}\,m}\mathbf{p} \tag{0.11}$$

The probability density is

$$P = \psi_{\mathbf{p}}^{*}\psi_{\mathbf{p}} = \frac{1}{(2\pi\hbar)^{3}} \tag{0.12}$$

Thus the current can be written as

$$\mathbf{j} = \frac{P}{m}\mathbf{p} \tag{0.13}$$

Classically, the momentum is $\mathbf{p} = mv$, so the current has the same form as $\mathbf{j} = P\mathbf{v}$. This is similar to the electromagnetic case where the electric current density $\mathbf{J} = \rho\mathbf{v}$ where ρ is the charge density and $\mathbf{v}$ is the velocity of that charge. The probability density can be viewed as "probability" moving with velocity $\mathbf{v}$.

Example 3. Now consider a one-dimensional problem where the wave function consists of two oppositely-moving plane waves:

$$\psi = Ae^{ipx/\hbar} + Be^{-ipx/\hbar} \tag{0.14}$$

In this case, we have

$$\frac{2mi}{\hbar}j = \psi^{*}\nabla\psi - \psi\nabla\psi^{*} \tag{0.15}$$

$$= \left(A^{*}e^{-ipx/\hbar} + B^{*}e^{ipx/\hbar}\right)\frac{ip}{\hbar}\left(Ae^{ipx/\hbar} + Be^{-ipx/\hbar}\right) -$$

$$\left(Ae^{ipx/\hbar} + Be^{-ipx/\hbar}\right)\frac{ip}{\hbar}\left(-A^{*}e^{-ipx/\hbar} + B^{*}e^{ipx/\hbar}\right) \tag{0.16}$$

$$= \frac{2ip}{\hbar}\left(|A|^{2} - |B|^{2}\right) \tag{0.17}$$

$$j = \frac{p}{m}\left(|A|^{2} - |B|^{2}\right) \tag{0.18}$$

The probability current separates into two terms, one for each direction of momentum.