HARMONIC OSCILLATOR - SERIES SOLUTION REVISITED

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercise 7.3.1.

Shankar's derivation of the eigenfunctions of the harmonic oscillator in the position basis is essentially the same as that in Griffiths, which we've covered before. The reader may wish to refresh their knowledge of this before reading the rest of this post.

To make the comparison we note that ϵ in Griffiths is 2ε in Shankar:

$$\varepsilon \equiv \frac{E}{\hbar\omega} \tag{1}$$

The analysis begins with the Schrödinger equation for the harmonic oscillator, which is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \tag{2}$$

Making the substitution

$$y\equiv\sqrt{\frac{m\omega}{\hbar}}x \tag{3}$$

we convert the equation to

$$\psi''+\left(2\varepsilon-y^2\right)\psi=0 \tag{4}$$

where a prime indicates a derivative with respect to y.

As explained in the earlier post, we further convert this equation by defining another function $u(y)$ (Griffiths calls this function $f(y)$) as

$$\psi(y)=e^{-y^2/2}u(y) \tag{5}$$

This results in a simpler differential equation for f:

$$\frac{d^2u}{dy^2}-2y\frac{du}{dy}+(2\varepsilon-1)u=0 \tag{6}$$

We can solve this by proposing that u is a power series in y:

$$u(y)=\sum_{n=0}^{\infty}C_ny^n \tag{7}$$

This leads to the recursion relation for the coefficients C_n:

$$C_{n+2}=C_n\frac{2n+1-2\varepsilon}{(n+1)(n+2)} \tag{8}$$

In order that u is finite for large y, this series must terminate, which leads to the quantization condition for the energy:

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right) \tag{9}$$

Shankar poses as an exercise the question as to why we didn't just try a series solution of 4, that is, we propose

$$\psi(y)=\sum_{n=0}^{\infty}A_ny^n \tag{10}$$

for some other coefficients A_n. If we try this, there are three terms with different exponents for y that result from plugging this into 4.

$$\psi'' = \sum_{n=0}^{\infty}A_nn(n-1)y^{n-2} \tag{11}$$

$$2\varepsilon\psi = 2\varepsilon\sum_{n=0}^{\infty}A_ny^n \tag{12}$$

$$-y^2\psi = -\sum_{n=0}^{\infty}A_ny^{n+2} \tag{13}$$

To compare the coefficients we reassign the summation ranges so that the powers of y are the same in all three terms.

$$\psi''=\sum_{n=2}^{\infty}A_nn(n-1)y^{n-2}=\sum_{n=0}^{\infty}A_{n+2}(n+2)(n+1)y^n \tag{14}$$

$$2\varepsilon\psi=2\varepsilon\sum_{n=0}^{\infty}A_ny^n \tag{15}$$

$$-y^2\psi=-\sum_{n=0}^{\infty}A_ny^{n+2}=-\sum_{n=2}^{\infty}A_{n-2}y^n \tag{16}$$

Note that the top two sums start at $n=0$ while the last sum starts at $n=2$. To satisfy 4, the coefficient of each power of y must be zero, that is

$$A_{n+2}(n+2)(n+1)+2\varepsilon A_n-A_{n-2}=0 \tag{17}$$

There are two separate conditions here; one for even n and the other for odd n. To get either sequence started, we need to specify the first two terms. For example, in the even sequence, we need to specify A_0 and A_2 which then allows calculation of A_4 (when $n=2$). We can then use A_2 and A_4 to get A_6 and so on. The general formula is

$$A_{n+2}=\frac{A_{n-2}-2\varepsilon A_n}{(n+2)(n+1)} \tag{18}$$

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