HARMONIC OSCILLATOR: HERMITE POLYNOMIALS AND ORTHOGONALITY OF EIGENFUNCTIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercises 7.3.2 - 7.3.3.

The eigenfunctions of the harmonic oscillator are given by

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-m\omega x^2/2\hbar} \tag{1}$$

where $H_n(u)$ is a Hermite polynomial. The Hermite polynomials obey the recursion relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2}$$

The first few Hermite polynomials are given in Shankar's equation 7.3.21, and we may use these to verify this relation for a couple of cases. Taking $n = 2$ we have

$$H_3(x) = 2xH_2(x) - 4H_1(x) \tag{3}$$
$$= 2x\left[-2\left(1-2x^2\right)\right] - 4(2x) \tag{4}$$
$$= -12x + 8x^3 \tag{5}$$

The last line agrees with H_3 as given in Shankar.

For $n = 3$ we have

$$H_4(x) = 2xH_3(x) - 6H_2(x) \tag{6}$$
$$= 2x\left[-12x + 8x^3\right] - 6\left[-2\left(1-2x^2\right)\right] \tag{7}$$
$$= 12 - 48x^2 + 16y^4 \tag{8}$$

which again agrees with Shankar's equation.

When deriving the solution in terms of Hermite polynomials, we followed Griffiths and found that we could write the polynomials in the form

$$H_n(y) = \sum_{j=0}^{n} a_j y^j \tag{9}$$

where the coefficients a_j obey the recursion relation

$$a_{j+2}=\frac{2j+1-\varepsilon}{(j+1)(j+2)}a_j \tag{10}$$

The ε used by Griffiths is equivalent to 2ε in Shankar, so using Shankar's notation, we see that this recursion relation is the same as Shankar's equation 7.3.15:

$$C_{n+2}=C_n\frac{2n+1-2\varepsilon}{(n+1)(n+2)} \tag{11}$$

Here, we have

$$\varepsilon=\frac{E}{\hbar\omega} \tag{12}$$

where E is the energy of the oscillator state.
Looking at the polynomials in Shankar's equation 7.3.21, we have

$$H_3(y)=-12\left(y-\frac{2}{3}y^3\right) \tag{13}$$

so

$$C_1 = -12 \tag{14}$$
$$C_3 = 8 \tag{15}$$

With $n=1$, we get from 11

$$C_3=-12\frac{3-2\varepsilon}{6} \tag{16}$$

However, for this state, $E=\left(3+\frac{1}{2}\right)\hbar\omega$ so $2\varepsilon=7$ and $C_3=8$ as required.
For H_4 we have

$$H_4(y)=12\left(1-4y^2+\frac{4}{3}y^4\right) \tag{17}$$

This means

$$C_0 = 12 \tag{18}$$
$$C_2 = -48 \tag{19}$$
$$C_4 = 16 \tag{20}$$

Here $E=\left(4+\frac{1}{2}\right)\hbar\omega$, so $2\varepsilon=9$ and

$$C_2 = 12\frac{(-8)}{2} = -48 \tag{21}$$

$$C_4 = -48\frac{5-9}{12} = 16 \tag{22}$$

We can see from the relation 2 that, given that $H_0 = 1$ and $H_1 = 2x$, all Hermite polynomials of even index contain only even powers of x, and all polynomials of odd index contain only odd powers of x. This means that all even Hermite polynomials are even functions of x, in the sense that $H_{2n}(-x) = H_{2n}(x)$, and all odd Hermite polynomials are odd functions of x, so that $H_{2n+1}(-x) = -H_{2n+1}(x)$.

If $\psi(x)$ is even and $\phi(x)$ is odd, then

$$\psi(-x)\phi(-x) = -\psi(x)\phi(x) \tag{23}$$

That is, the product $\psi(x)\phi(x)$ is an odd function. Since the integral of any odd function over an interval symmetric about $x = 0$ is zero, we have

$$\int_{-\infty}^{\infty} \psi(x)\phi(x)\,dx = 0 \tag{24}$$

Looking at the eigenfunctions 1, we see that the exponential factor is a Gaussian centred at $x = 0$ and is therefore even, so that ψ_n will be even or odd depending on whether n is even or odd. In particular, the integral of any even ψ_n multiplied by any odd ψ_n over all x will be zero.

To show that pairs of even functions are also orthogonal is a bit trickier, but we can do it in the simplest case, where we consider the functions ψ_0 and ψ_2.

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)\,dx = \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{8}}\int_{-\infty}^{\infty} H_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/\hbar}dx \tag{25}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{8}}\int_{-\infty}^{\infty} (1)\left[-2\left(1-2\frac{m\omega}{\hbar}x^2\right)\right] e^{-m\omega x^2/\hbar}dx \tag{26}$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2}}\left[\sqrt{\frac{\pi\hbar}{m\omega}} - \sqrt{\frac{\pi\hbar}{m\omega}}\right] \tag{27}$$

$$= 0 \tag{28}$$

The two Gaussian integrals can be done using standard formulas as given in Shankar's Appendix A.2. (I used Maple.)

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