# HARMONIC OSCILLATOR: MATRIX ELEMENTS USING HERMITE POLYNOMIALS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercise 7.3.4.

Earlier, we found the matrix elements of $X$ and $P$ of the harmonic oscillator using the raising and lowering operators. We can also find these matrix elements using the recursion relations and orthogonality of Hermite polynomials. The required relations are also given as Shankar's equations 7.3.24 - 7.3.26.

$$H_n'(y) = 2nH_{n-1}(y) \tag{1}$$

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) \tag{2}$$

$$\int_{-\infty}^{\infty} H_n(y)H_{n'}(y)e^{-y^2}dy = \sqrt{\pi}2^n n!\delta_{nn'} \tag{3}$$

The energy eigenfunctions of the harmonic oscillator are

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-y^2/2} \tag{4}$$

where

$$y \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{5}$$

The matrix elements of $x$ are therefore

$$\left\langle n'\left|X\right|n\right\rangle = \int_{-\infty}^{\infty}\psi_{n'}(x)\,x\,\psi_n(x)\,dx \tag{6}$$

where we've used the fact that $\psi_{n'}(x)$ is real, so its complex conjugate is the same as the original. Converting to the variable $y$ using 5 we have

(7)
$$\left\langle n'\left|X\right|n\right\rangle = \frac{\hbar}{m\omega}\int_{-\infty}^{\infty}\psi_{n'}(y)\,y\psi_{n}(y)\,dy$$

(8)
$$= \frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2^{n'}2^{n}n'!n!}}\int_{-\infty}^{\infty}H_{n}(y)\,yH_{n'}(y)\,e^{-y^{2}}dy$$

(9)
$$= \sqrt{\frac{\hbar}{\pi m\omega}}\frac{1}{\sqrt{2^{n'}2^{n}n'!n!}}\int_{-\infty}^{\infty}\frac{1}{2}\left[H_{n+1}+2nH_{n-1}\right]H_{n'}e^{-y^{2}}dy$$

(10)
$$= \sqrt{\frac{\hbar}{m\omega}}\frac{2^{n'}n'!}{2\sqrt{2^{n'}2^{n}n'!n!}}\left[\delta_{n',n+1}+2n\delta_{n',n-1}\right]$$

(11)
$$= \sqrt{\frac{\hbar}{m\omega}}\left[\frac{2^{n+1}(n+1)!}{2\sqrt{2^{2n+1}(n+1)!n!}}\delta_{n',n+1}+\frac{2^{n}n(n-1)!}{2\sqrt{2^{2n-1}(n-1)!n!}}\delta_{n',n-1}\right]$$

(12)
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\frac{(n+1)\,n!}{\sqrt{(n+1)\,(n!)^{2}}}\delta_{n',n+1}+\frac{n\,(n-1)!}{\sqrt{n\left[(n-1)!\right]^{2}}}\delta_{n',n-1}\right]$$

(13)
$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\delta_{n',n+1}+\sqrt{n}\delta_{n',n-1}\right]$$

We used 2 to get the third line and 3 to do the integrals.

For the matrix elements of $P$ we use

(14)
$$P = -i\hbar\frac{d}{dx} = -i\sqrt{\hbar m\omega}\frac{d}{dy}$$

(15)
$$P\psi_{n}(y) = -i\sqrt{\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{n}n!}}e^{-y^{2}/2}\left[H_{n}'(y)-yH_{n}(y)\right]$$

(16)
$$= -i\sqrt{\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{n}n!}}e^{-y^{2}/2}\left[2nH_{n-1}-\frac{1}{2}\left(H_{n+1}+2nH_{n-1}\right)\right]$$

(17)
$$= -i\sqrt{\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{n}n!}}e^{-y^{2}/2}\left[nH_{n-1}-\frac{1}{2}H_{n+1}\right]$$

We used 2 to get the third line.
We get, using $dx=\sqrt{\hbar/m\omega}dy$ from 5

(18)
$$\left\langle n'\left|P\right|n\right\rangle = -i\sqrt{\hbar m\omega}\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2^{n'}n'!2^{n}n!}}\int_{-\infty}^{\infty}H_{n'}\left[nH_{n-1}-\frac{1}{2}H_{n+1}\right]e^{-y^{2}}dy$$
(19)
$$= -i\sqrt{\hbar m\omega}\left[\frac{2^{n'}n'!n}{\sqrt{2^{n'}n'!2^{n}n!}}\delta_{n',n-1}-\frac{2^{n'-1}n'!}{\sqrt{2^{n'}n'!2^{n}n!}}\delta_{n',n+1}\right]$$
(20)
$$= -i\sqrt{\hbar m\omega}\left[\frac{2^{n-1}\left(n-1\right)!n}{\sqrt{2^{n-1}\left(n-1\right)!2^{n}n!}}\delta_{n',n-1}-\frac{2^{n}\left(n+1\right)!}{\sqrt{2^{n+1}\left(n+1\right)!2^{n}n!}}\delta_{n',n+1}\right]$$
(21)
$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\delta_{n',n+1}-\sqrt{n}\delta_{n',n-1}\right]$$

## PINGBACKS

Pingback: Harmonic oscillator - mean position and momentum