HARMONIC OSCILLATOR - MEAN POSITION AND MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercise 7.3.5.

The energy eigenfunctions of the harmonic oscillator are

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2} \tag{1}$$

where

$$y \equiv \sqrt{\frac{m\omega}{\hbar}} x \tag{2}$$

and H_n is the Hermite polynomial of order n. Since an even (odd) Hermite polynomial is an even (odd) function, ψ_n is even (odd) if n is even (odd), so we can use this fact to show that

$$\langle n|X|n\rangle = \int_{-\infty}^{\infty} x\psi_n^2(x)\, dx = 0 \tag{3}$$

This follows because the square of either an even or odd function gives an even function, and x itself is odd, so the integrand is the product of an odd and even function, which is odd. The integral over any interval symmetric about $x=0$ of an odd function is odd. Thus the mean position $\langle X\rangle$ of a particle in any of the harmonic oscillator's energy eigenstates is zero.

For the momentum P, we have

$$\langle n|P|n\rangle = -i\hbar \int_{-\infty}^{\infty} \psi_n \frac{d\psi_n}{dx}\, dx \tag{4}$$

As we showed earlier

$$-i\hbar\frac{d\psi_n}{dx} = i\sqrt{\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2}\left[nH_{n-1} - \frac{1}{2}H_{n+1}\right] \tag{5}$$

If n is even (odd), then $nH_{n-1}-\frac{1}{2}H_{n+1}$ is odd (even), so the product $\psi_n\frac{d\psi_n}{dx}$ is always the product of one odd and one even function, making it odd. Thus

$$\langle n|P|n\rangle = 0 \tag{6}$$

Thus the mean momentum $\langle P\rangle = 0$ in all energy eigenstates.

This means that the uncertainties in position and momentum are determined entirely by the mean square values:

$$(\Delta X)^2 = \left\langle X^2\right\rangle - \langle X\rangle^2 = \left\langle X^2\right\rangle \tag{7}$$

$$(\Delta P)^2 = \left\langle P^2\right\rangle \tag{8}$$

We can work out these values for a couple of specific states. For $n=1$ we have

$$\left\langle 1\left|X^2\right|1\right\rangle = \int_{-\infty}^{\infty} x^2\psi_1^2(x)\;dx \tag{9}$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} x^2 H_1^2(y)\,e^{-y^2}dx \tag{10}$$

$$= \frac{1}{2}\frac{\hbar}{\sqrt{\pi}m\omega}\int_{-\infty}^{\infty} y^2 H_1^2(y)\,e^{-y^2}dy \tag{11}$$

$$= \frac{2\hbar}{\sqrt{\pi}m\omega}\int_{-\infty}^{\infty} y^4 e^{-y^2}dy \tag{12}$$

$$= \frac{2\hbar}{\sqrt{\pi}m\omega}\frac{3\sqrt{\pi}}{4} \tag{13}$$

$$= \frac{3\hbar}{2m\omega} \tag{14}$$

We've used

$$H_1(y) = 2y \tag{15}$$

and formula just before A.2.3 from the appendix in Shankar, which gives

$$I_4(\alpha) = \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx \tag{16}$$

$$= \frac{\partial^2}{\partial\alpha^2} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \tag{17}$$

$$= \frac{\partial^2}{\partial\alpha^2} I_0(\alpha) \tag{18}$$

From formula A.2.2

$$I_4(\alpha) = \frac{\partial^2}{\partial\alpha^2} \sqrt{\frac{\pi}{\alpha}} \tag{19}$$

$$= \frac{3\sqrt{\pi}}{4\alpha^{5/2}} \tag{20}$$

Setting $\alpha = 1$ gives

$$\int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3\sqrt{\pi}}{4} \tag{21}$$

For P, we have

$$\left\langle 1 \left| P^2 \right| 1 \right\rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_1 \frac{d^2}{dx^2} \psi_1 \, dx \tag{22}$$

The derivative is

$$\frac{d^2}{dx^2}\psi_1 = \frac{d^2\psi_1}{dy^2} \left(\frac{dy}{dx}\right)^2 \tag{23}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} \left[H_1(y) e^{-y^2/2}\right] \tag{24}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} \left[2y e^{-y^2/2}\right] \tag{25}$$

$$= \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} \frac{m\omega}{\hbar} e^{-y^2/2} \left[2y^3 - 6y\right] \tag{26}$$

We can now evaluate 22:

$$-\hbar^2\int_{-\infty}^{\infty}\psi_1\frac{d^2}{dx^2}\psi_1\,dx=-\hbar^2\sqrt{\frac{m\omega}{4\pi\hbar}}\frac{m\omega}{\hbar}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}2ye^{-y^2}2\left[y^3-3y\right]dy \tag{27}$$

$$=-\frac{2m\omega\hbar}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-y^2}\left[y^4-3y^2\right]dy \tag{28}$$

$$=-\frac{2m\omega\hbar}{\sqrt{\pi}}\left[\frac{3\sqrt{\pi}}{4}-\frac{3\sqrt{\pi}}{2}\right] \tag{29}$$

$$=\frac{3}{2}m\omega\hbar \tag{30}$$

Thus for the $n=1$ state

$$\Delta X = \sqrt{\frac{3\hbar}{2m\omega}} \tag{31}$$

$$\Delta P = \sqrt{\frac{3}{2}m\omega\hbar} \tag{32}$$

$$\Delta X\Delta P = \frac{3}{2}\hbar > \frac{\hbar}{2} \tag{33}$$

For the $n=0$ (ground) state, we can use $H_0=1$ to get

$$\left\langle X^2\right\rangle=\int_{-\infty}^{\infty}x^2\psi_0^2(x)\,dx \tag{34}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}x^2H_0^2(y)\,e^{-y^2}dx \tag{35}$$

$$=\frac{\hbar}{\sqrt{\pi}m\omega}\int_{-\infty}^{\infty}y^2e^{-y^2}dy \tag{36}$$

$$=\frac{\hbar}{2m\omega} \tag{37}$$

For P:

$$\frac{d^2}{dx^2}\psi_0 = \frac{d^2\psi_0}{dy^2}\left(\frac{dy}{dx}\right)^2 \tag{38}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{m\omega}{\hbar}e^{-y^2/2}\left(y^2-1\right) \tag{39}$$

(40)

$$\left\langle P^2\right\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi_0\frac{d^2}{dx^2}\psi_0\,dx = -\hbar^2\sqrt{\frac{m\omega}{\pi\hbar}}\frac{m\omega}{\hbar}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}e^{-y^2}\left(y^2-1\right)dy$$

$$= -\frac{m\omega\hbar}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}-\sqrt{\pi}\right] \tag{41}$$

$$= \frac{1}{2}m\omega\hbar \tag{42}$$

The uncertainty principle in this case gives

$$\Delta X\Delta P = \frac{\hbar}{2} \tag{43}$$

so it saturates the condition $\Delta X\Delta P \geq \frac{\hbar}{2}$.

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