

## HARMONIC OSCILLATOR - MEAN POSITION AND MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.

Section 7.3, Exercise 7.3.5.

The energy eigenfunctions of the harmonic oscillator are

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2} \quad (1)$$

where

$$y \equiv \sqrt{\frac{m\omega}{\hbar}} x \quad (2)$$

and  $H_n$  is the Hermite polynomial of order  $n$ . Since an even (odd) Hermite polynomial is an even (odd) function,  $\psi_n$  is even (odd) if  $n$  is even (odd), so we can use this fact to show that

$$\langle n|X|n\rangle = \int_{-\infty}^{\infty} x \psi_n^2(x) dx = 0 \quad (3)$$

This follows because the square of either an even or odd function gives an even function, and  $x$  itself is odd, so the integrand is the product of an odd and even function, which is odd. The integral over any interval symmetric about  $x = 0$  of an odd function is zero. Thus the mean position  $\langle X \rangle$  of a particle in any of the harmonic oscillator's energy eigenstates is zero.

For the momentum  $P$ , we have

$$\langle n|P|n\rangle = -i\hbar \int_{-\infty}^{\infty} \psi_n \frac{d\psi_n}{dx} dx \quad (4)$$

As we showed earlier

$$-i\hbar \frac{d\psi_n}{dx} = i\sqrt{\hbar m\omega} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2} \left[ nH_{n-1} - \frac{1}{2}H_{n+1} \right] \quad (5)$$

If  $n$  is even (odd), then  $nH_{n-1} - \frac{1}{2}H_{n+1}$  is odd (even), so the product  $\psi_n \frac{d\psi_n}{dx}$  is always the product of one odd and one even function, making it odd. Thus

$$\langle n|P|n\rangle = 0 \quad (6)$$

Thus the mean momentum  $\langle P\rangle = 0$  in all energy eigenstates.

This means that the uncertainties in position and momentum are determined entirely by the mean square values:

$$(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2 = \langle X^2\rangle \quad (7)$$

$$(\Delta P)^2 = \langle P^2\rangle \quad (8)$$

We can work out these values for a couple of specific states. For  $n = 1$  we have

$$\langle 1|X^2|1\rangle = \int_{-\infty}^{\infty} x^2 \psi_1^2(x) dx \quad (9)$$

$$= \frac{1}{2} \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 H_1^2(y) e^{-y^2} dx \quad (10)$$

$$= \frac{1}{2} \frac{\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^2 H_1^2(y) e^{-y^2} dy \quad (11)$$

$$= \frac{2\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^4 e^{-y^2} dy \quad (12)$$

$$= \frac{2\hbar}{\sqrt{\pi m\omega}} \frac{3\sqrt{\pi}}{4} \quad (13)$$

$$= \frac{3\hbar}{2m\omega} \quad (14)$$

We've used

$$H_1(y) = 2y \quad (15)$$

and formula just before A.2.3 from the appendix in Shankar, which gives

$$I_4(\alpha) = \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx \quad (16)$$

$$= \frac{\partial^2}{\partial \alpha^2} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \quad (17)$$

$$= \frac{\partial^2}{\partial \alpha^2} I_0(\alpha) \quad (18)$$

From formula A.2.2

$$I_4(\alpha) = \frac{\partial^2}{\partial \alpha^2} \sqrt{\frac{\pi}{\alpha}} \quad (19)$$

$$= \frac{3\sqrt{\pi}}{4\alpha^{5/2}} \quad (20)$$

Setting  $\alpha = 1$  gives

$$\int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3\sqrt{\pi}}{4} \quad (21)$$

For  $P$ , we have

$$\langle 1 | P^2 | 1 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_1 \frac{d^2}{dx^2} \psi_1 dx \quad (22)$$

The derivative is

$$\frac{d^2}{dx^2} \psi_1 = \frac{d^2 \psi_1}{dy^2} \left( \frac{dy}{dx} \right)^2 \quad (23)$$

$$= \frac{1}{\sqrt{2}} \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} [H_1(y) e^{-y^2/2}] \quad (24)$$

$$= \frac{1}{\sqrt{2}} \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} [2ye^{-y^2/2}] \quad (25)$$

$$= \left( \frac{m\omega}{4\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} e^{-y^2/2} [2y^3 - 6y] \quad (26)$$

We can now evaluate 22:

$$-\hbar^2 \int_{-\infty}^{\infty} \psi_1 \frac{d^2}{dx^2} \psi_1 dx = -\hbar^2 \sqrt{\frac{m\omega}{4\pi\hbar}} \frac{m\omega}{\hbar} \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} 2ye^{-y^2/2} [y^3 - 3y] dy \quad (27)$$

$$= -\frac{2m\omega\hbar}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} [y^4 - 3y^2] dy \quad (28)$$

$$= -\frac{2m\omega\hbar}{\sqrt{\pi}} \left[ \frac{3\sqrt{\pi}}{4} - \frac{3\sqrt{\pi}}{2} \right] \quad (29)$$

$$= \frac{3}{2} m\omega\hbar \quad (30)$$

Thus for the  $n = 1$  state

$$\Delta X = \sqrt{\frac{3\hbar}{2m\omega}} \quad (31)$$

$$\Delta P = \sqrt{\frac{3}{2}m\omega\hbar} \quad (32)$$

$$\Delta X \Delta P = \frac{3}{2}\hbar > \frac{\hbar}{2} \quad (33)$$

For the  $n = 0$  (ground) state, we can use  $H_0 = 1$  to get

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi_0^2(x) dx \quad (34)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 H_0^2(y) e^{-y^2} dx \quad (35)$$

$$= \frac{\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy \quad (36)$$

$$= \frac{\hbar}{2m\omega} \quad (37)$$

For  $P$ :

$$\frac{d^2}{dx^2} \psi_0 = \frac{d^2 \psi_0}{dy^2} \left( \frac{dy}{dx} \right)^2 \quad (38)$$

$$= \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} e^{-y^2/2} (y^2 - 1) \quad (39)$$

$$\langle P^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_0 \frac{d^2}{dx^2} \psi_0 dx = -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \frac{m\omega}{\hbar} \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} e^{-y^2} (y^2 - 1) dy \quad (40)$$

$$= -\frac{m\omega\hbar}{\sqrt{\pi}} \left[ \frac{\sqrt{\pi}}{2} - \sqrt{\pi} \right] \quad (41)$$

$$= \frac{1}{2} m\omega\hbar \quad (42)$$

The uncertainty principle in this case gives

$$\Delta X \Delta P = \frac{\hbar}{2} \quad (43)$$

so it saturates the condition  $\Delta X \Delta P \geq \frac{\hbar}{2}$ .

PINGBACKS

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