HARMONIC OSCILLATOR - EIGENFUNCTIONS IN MOMENTUM SPACE

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercise 7.3.7.

We've seen how to solve the Schrödinger equation for the harmonic oscillator in the position basis, where the independent variable is x. It's actually fairly easy to adapt this solution to find the wave functions in momentum space. (We've also found these functions by using the Fourier transform of the position functions, but the present post shows an easier way.)

The Schrödinger equation for the stationary states of the harmonic oscillator is, in operator form:

$$\frac{P^2}{2m}\psi+\frac{1}{2}m\omega^2X^2\psi=E\psi \tag{1}$$

To work in momentum space, we use the results

$$P = p \tag{2}$$

$$X = i\hbar\frac{\partial}{\partial p} \tag{3}$$

This gives

$$\frac{p^2}{2m}\psi-\frac{1}{2}\hbar^2m\omega^2\frac{d^2\psi}{dp^2}=E\psi \tag{4}$$

Dividing through by $(m\omega)^2$ we get

$$-\frac{\hbar^2}{2m}\psi''+\frac{p^2}{2m^3\omega^2}=\frac{E}{(m\omega)^2}\psi \tag{5}$$

where a prime on ψ indicates a derivative with respect to p.

This is similar to the Schrödinger equation in position space:

$$-\frac{\hbar^2}{2m}\psi''+\frac{1}{2}m\omega^2x^2\psi=E\psi \tag{6}$$

(where a prime here indicates a derivative with respect to x). When we solved the position space equation, we introduced a dimensionless variable

$$y \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{7}$$

Using this technique to solve 5, we try a definition for y of

$$y \equiv \frac{p}{\sqrt{\hbar m\omega}} \tag{8}$$

(You can check the units of $\sqrt{\hbar m\omega}$ to see they are the units of momentum, so y is indeed dimensionless here.) Making this substitution, we get

$$\frac{d^2\psi}{dp^2} = \frac{1}{\hbar m\omega}\frac{d^2\psi}{dy^2} \tag{9}$$

$$\frac{p^2}{2m^3\omega^2} = \frac{\hbar y^2}{2m^2\omega} \tag{10}$$

Thus 5 becomes

$$-\frac{\hbar^2}{2m}\frac{1}{\hbar m\omega}\frac{d^2\psi}{dy^2} + \frac{\hbar y^2}{2m^2\omega}\psi = \frac{E}{(m\omega)^2}\psi \tag{11}$$

$$\frac{\hbar^2}{2m}\left[-\frac{d^2\psi}{dy^2} + y^2\psi\right] = \frac{\hbar E}{\omega}\psi \tag{12}$$

We can now use the same dimensionless parameter we used in the earlier derivation:

$$\varepsilon \equiv \frac{E}{\hbar\omega} \tag{13}$$

This results in the differential equation

$$\psi'' + \left(2\varepsilon - y^2\right)\psi = 0 \tag{14}$$

where a prime now indicates a derivative with respect to y. This is exactly the same differential equation that we got for the position basis, except that the independent variable y is now defined in terms of p by 8 instead of x. We can solve it in the same way, which results in the same quantization condition on the allowable energies of $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. The eigenfunctions look the same when expressed in terms of y:

$$\psi_n(y) = A\frac{1}{\sqrt{2^n n!}}H_n(y)e^{-y^2/2} \tag{15}$$

where A is a normalization constant with the value in the position basis of

$$A=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \tag{16}$$

and H_n is a Hermite polynomial. We can get the eigenfunctions in momentum space by replacing y by 8. We can see that this amounts to replacing $x\to p$ and $m\omega\to\frac{1}{m\omega}$, so we get

$$\psi_n(p)=\frac{1}{(\pi\hbar m\omega)^{1/4}}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right)e^{-p^2/2\hbar m\omega} \tag{17}$$

In particular, the ground state is

$$\psi_0(p)=\frac{1}{(\pi\hbar m\omega)^{1/4}}e^{-p^2/2\hbar m\omega} \tag{18}$$

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