VIRIAL THEOREM IN CLASSICAL MECHANICS; APPLICATION TO HARMONIC OSCILLATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.3.

We've seen the virial theorem in quantum mechanics, but this theorem was originally devised in classical mechanics. For a single particle, we consider the quantity

$$G = \mathbf{r}\cdot\mathbf{p} \tag{1}$$

that is, the product of position and momentum. Taking the time derivative, we have

$$\begin{aligned} \frac{dG}{dt} &= \mathbf{p}\cdot\frac{d\mathbf{r}}{dt}+\mathbf{r}\cdot\frac{d\mathbf{p}}{dt} && (2)\\ &= mv^2+\mathbf{r}\cdot\mathbf{F} && (3) \end{aligned}$$

where $\mathbf{v}=d\mathbf{r}/dt$ is the velocity of the particle, $\mathbf{p}=m\mathbf{v}$ and $\mathbf{F}=d\mathbf{p}/dt$ is the force acting on the particle. If the force is a central force (that is, it depends only on the particle's distance from some centre point $\mathbf{r}=0$, then the force can be written as the negative gradient of a potential V that depends only on $\mathbf{r}$. In the case where V depends only on a power of r, we have

$$\begin{aligned} V &= ar^k && (4)\\ \mathbf{F} &= -\frac{dV}{dr}=-kar^{k-1}\hat{\mathbf{r}} && (5) \end{aligned}$$

In that case, we have

$$\begin{aligned} \frac{dG}{dt} &= mv^2-\mathbf{r}\cdot kar^{k-1}\hat{\mathbf{r}} && (6)\\ &= 2T-kar^k && (7)\\ &= 2T-kV && (8) \end{aligned}$$

where T is the kinetic energy $T = \frac{1}{2}mv^2$. If the particle is moving in a circular orbit then its average position and average momentum (averaged over one orbit) do not change with time, so $\frac{dG}{dt} = 0$ and we get

$$2\langle T\rangle - k\langle V\rangle = 0 \tag{9}$$

$$\langle T\rangle = \frac{k}{2}\langle V\rangle \tag{10}$$

Another way of seeing this is that, in a circular orbit at constant orbital speed, the only force acting is the centripetal force holding the particle in its orbit, which is

$$F_{cen} = -\frac{mv^2}{r} \tag{11}$$

where the minus sign indicates that the force acts in the opposite direction to the outward pointing radius vector.

This force is provided by the gradient of the potential, so we have

$$F_{cen} = -\frac{dV}{dr} = -kar^{k-1} \tag{12}$$

We therefore have

$$\frac{mv^2}{r} = kar^{k-1} \tag{13}$$

$$mv^2 = 2T = kar^k = kV \tag{14}$$

$$\langle T\rangle = \frac{k}{2}\langle V\rangle \tag{15}$$

For the case of a harmonic oscillator, $V = \frac{1}{2}m\omega^2 x^2$ so the exponent is $k = 2$ and we have $T = V$. We can verify this by calculating the mean kinetic and potential energies explicitly, using earlier results. In the oscillator state $|n\rangle$ we have

$$\langle x^2\rangle = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \tag{16}$$

$$\langle p^2\rangle = \hbar m\omega\left(n+\frac{1}{2}\right) \tag{17}$$

The energies are

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) \tag{18}$$

$$\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) \tag{19}$$

Therefore $\langle T\rangle = \langle V\rangle$ as required.