HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATORS AS FUNCTIONS OF TIME

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.6.

We'll consider here the problem of finding the averages of the raising and lowering operators (from the harmonic oscillator) as functions of time, that is, we want to find $\langle a(t)\rangle$ and $\left\langle a^{\dagger}(t)\right\rangle$. At first glance we might think they are both zero, since they are defined in terms of position and momentum as

$$a^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}\left[-iP+m\omega X\right] \tag{1}$$

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left[iP+m\omega X\right] \tag{2}$$

and the averages of P and X in any of the energy eigenstates of the harmonic oscillator are all zero. However, suppose we have a mixed state $|\psi\rangle$ which can be written as a sum over the eigenstates as

$$\psi(t) = \sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle \tag{3}$$

$$= \sum_{n=0}^{\infty} c_n e^{-i(2n+1)\omega t/2}|n\rangle \tag{4}$$

where in the second line we used the energies of the oscillator as

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \tag{5}$$

We now have

$$\langle a(t)\rangle = \langle\psi|a|\psi\rangle \tag{6}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* e^{i(2m+1)\omega t/2} c_n e^{-i(2n+1)\omega t/2}\langle m|a|n\rangle \tag{7}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\langle m|a|n\rangle \tag{8}$$

We can now use the formula

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{9}$$

This gives

$$\langle a(t)\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\sqrt{n}\langle m|n-1\rangle \tag{10}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\sqrt{n}\delta_{m,n-1} \tag{11}$$

$$= e^{-i\omega t}\sum_{n=0}^{\infty} c_{n-1}^* c_n\sqrt{n} \tag{12}$$

$$= e^{-i\omega t}\langle a(0)\rangle \tag{13}$$

Note that if $|\psi\rangle$ is an eigenstate, then only one of the coefficients c_n is non-zero, so $\langle a(0)\rangle = 0$ as we'd expect.

The derivation for $\left\langle a^\dagger(t)\right\rangle$ is similar:

$$\left\langle a^\dagger(t)\right\rangle = \left\langle \psi\left|a^\dagger\right|\psi\right\rangle \tag{14}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* e^{i(2m+1)\omega t/2} c_n e^{-i(2n+1)\omega t/2}\left\langle m\left|a^\dagger\right|n\right\rangle \tag{15}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\left\langle m\left|a^\dagger\right|n\right\rangle \tag{16}$$

We can now use the formula

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{17}$$

This gives

$$\left\langle a^\dagger(t)\right\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\sqrt{n+1}\langle m|n+1\rangle \tag{18}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n e^{i(m-n)\omega t}\sqrt{n+1}\delta_{m,n+1} \tag{19}$$

$$= e^{i\omega t}\sum_{n=0}^{\infty} c_{n+1}^* c_n\sqrt{n+1} \tag{20}$$

$$= e^{i\omega t}\left\langle a^\dagger(0)\right\rangle \tag{21}$$