

## POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlier included specifications for the matrix elements of position  $X$  and momentum  $P$  in position space:

$$\langle x|X|x'\rangle = x\delta(x-x') \quad (1)$$

$$\langle x|P|x'\rangle = -i\hbar\delta'(x-x') \quad (2)$$

A more fundamental form of this postulate is to specify the commutation relation between  $X$  and  $P$ , which is independent of the basis and is

$$[X, P] = i\hbar \quad (3)$$

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$X = i\hbar\frac{d}{dp} \quad (4)$$

$$P = p \quad (5)$$

We can verify this by calculating the commutator by applying it to a function  $f(p)$ :

$$[X, P]f = i\hbar\frac{d}{dp}(pf(p)) - i\hbar p\frac{d}{dp}f(p) \quad (6)$$

$$= i\hbar f(p) + i\hbar p\frac{d}{dp}f(p) - i\hbar p\frac{d}{dp}f(p) \quad (7)$$

$$= i\hbar f(p) \quad (8)$$

Thus 3 is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracket for two physical quantities in the classical system, which gives

$$\{\omega, \lambda\} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \quad (9)$$

where  $q_i$  and  $p_i$  are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by  $i\hbar$  times the corresponding commutator. That is

$$[\Omega, \Lambda] = i\hbar \{\omega, \lambda\} \quad (10)$$

For the case of  $X$  and  $P$ , we have, in classical mechanics in one dimension

$$\{x, p\} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} = 1 \quad (11)$$

so the quantum commutator is given by 3.

For other quantities, we can use the theorems on the Poisson bracketsto reduce them:

$$\{\omega, \lambda\} = -\{\lambda, \omega\} \quad (12)$$

$$\{\omega, \lambda + \sigma\} = \{\omega, \lambda\} + \{\omega, \sigma\} \quad (13)$$

$$\{\omega, \lambda\sigma\} = \{\omega, \lambda\}\sigma + \{\omega, \sigma\}\lambda \quad (14)$$

Quantum commutators obey similar rules

$$[\Omega, \Lambda] = -[\Lambda, \Omega] \quad (15)$$

$$[\Omega, \Lambda + \Gamma] = [\Omega, \Lambda] + [\Omega, \Gamma] \quad (16)$$

$$[\Omega\Lambda, \Gamma] = \Omega[\Lambda, \Gamma] + [\Omega, \Gamma]\Lambda \quad (17)$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 14 we could also have written

$$\{\omega, \lambda\sigma\} = \sigma\{\omega, \lambda\} + \lambda\{\omega, \sigma\} \quad (18)$$

since all three quantities are numerical (not operators), so multiplication commutes. In 17 it is *not* true in general that, for example

$$\Omega[\Lambda, \Gamma] + [\Omega, \Gamma]\Lambda = [\Lambda, \Gamma]\Omega + [\Omega, \Gamma]\Lambda \quad (19)$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$\{\omega(x, p), \lambda(x, p)\} = \gamma(x, p) \quad (20)$$

by

$$[\Omega(X, P), \Lambda(X, P)] = i\hbar\Gamma(X, P) \quad (21)$$

where each of the operators in the last equation is obtained by replacing  $x$  in the first equation by  $X$  and  $p$  by  $P$ . We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

$$\Omega = X \quad (22)$$

$$\Lambda = X^2 + P^2 \quad (23)$$

In the classical version, we calculate the Poisson bracket

$$\{\omega, \lambda\} = \{x, x^2 + p^2\} \quad (24)$$

$$= \{x, x^2\} + \{x, p^2\} \quad (25)$$

$$= 0 + 2\{x, p\}p \quad (26)$$

$$= 2p \quad (27)$$

Thus, by our rule above, the quantum version should be

$$[\Omega, \Lambda] = 2i\hbar P \quad (28)$$

We can verify this using 17

$$[X, X^2 + P^2] = [X, X^2] + [X, P^2] \quad (29)$$

$$= 0 - [P^2, X] \quad (30)$$

$$= -P[P, X] - [P, X]P \quad (31)$$

$$= -P(-i\hbar) - (-i\hbar)P \quad (32)$$

$$= 2i\hbar P \quad (33)$$

In this case, there is no ordering ambiguity in the quantum version, since  $[X, P] = i\hbar$  is just a number.

For a second example, suppose we have

$$\Omega = X^2 \quad (34)$$

$$\Lambda = P^2 \quad (35)$$

The classical version gives us, using the relations 14, 11 and 27

$$\{x^2, p^2\} = -\{p^2, x^2\} \quad (36)$$

$$= -2\{p^2, x\}x \quad (37)$$

$$= 2\{x, p^2\}x \quad (38)$$

$$= 4px \quad (39)$$

In the classical case, this result is the same as  $4xp$ , but because  $X$  and  $P$  don't commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

$$[X^2, P^2] = X[X, P^2] + [X, P^2]X \quad (40)$$

From 33 we have

$$[X, P^2] = 2i\hbar P \quad (41)$$

so we get

$$[X^2, P^2] = 2i\hbar(XP + PX) \quad (42)$$

Thus if the Poisson bracket involves a product of  $p$  and  $x$ , this should be replaced by

$$xp \text{ or } px \rightarrow \frac{1}{2}(XP + PX) \quad (43)$$

in the quantum version.

#### PINGBACKS

Pingback: Angular momentum - Poisson bracket to commutator

Pingback: Hamiltonian in non-rectangular coordinates

Pingback: Harmonic oscillator: momentum space functions and Hermite polynomial recursion

Pingback: Direct product of two vector spaces

Pingback: Nonrelativistic field theory - Schrödinger equation