# POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlierincluded specifications for the matrix elements of position $X$ and momentum $P$ in position space:

$$\left\langle x\left|X\right|x'\right\rangle = x\delta\left(x-x'\right) \tag{1}$$

$$\left\langle x\left|P\right|x'\right\rangle = -i\hbar\delta'\left(x-x'\right) \tag{2}$$

A more fundamental form of this postulate is to specify the commutation relation between $X$ and $P$, which is independent of the basis and is

$$[X,P]=i\hbar \tag{3}$$

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$X = i\hbar\frac{d}{dp} \tag{4}$$

$$P = p \tag{5}$$

We can verify this by calculating the commutator by applying it to a function $f(p)$:

$$[X,P]f = i\hbar\frac{d}{dp}(pf(p))-i\hbar p\frac{d}{dp}f(p) \tag{6}$$

$$= i\hbar f(p)+i\hbar p\frac{d}{dp}f(p)-i\hbar p\frac{d}{dp}f(p) \tag{7}$$

$$= i\hbar f(p) \tag{8}$$

Thus 3 is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracketfor two physical quantities in the classical system, which gives

$$\{\omega,\lambda\}=\sum_{i}\left(\frac{\partial\omega}{\partial q_{i}}\frac{\partial\lambda}{\partial p_{i}}-\frac{\partial\omega}{\partial p_{i}}\frac{\partial\lambda}{\partial q_{i}}\right) \tag{9}$$

where $q_i$ and $p_i$ are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by $i\hbar$ times the corresponding commutator. That is

$$[\Omega,\Lambda]=i\hbar\{\omega,\lambda\} \tag{10}$$

For the case of $X$ and $P$, we have, in classical mechanics in one dimension

$$\{x,p\}=\frac{\partial x}{\partial x}\frac{\partial p}{\partial p}-\frac{\partial x}{\partial p}\frac{\partial p}{\partial x}=1 \tag{11}$$

so the quantum commutator is given by 3.

For other quantities, we can use the theorems on the Poisson bracketsto reduce them:

$$\begin{aligned}
\{\omega,\lambda\} &= -\{\lambda,\omega\} && (12)\\
\{\omega,\lambda+\sigma\} &= \{\omega,\lambda\}+\{\omega,\sigma\} && (13)\\
\{\omega,\lambda\sigma\} &= \{\omega,\lambda\}\sigma+\{\omega,\sigma\}\lambda && (14)
\end{aligned}$$

Quantum commutators obey similar rules

$$\begin{aligned}
[\Omega,\Lambda] &= -[\Lambda,\Omega] && (15)\\
[\Omega,\Lambda+\Gamma] &= [\Omega,\Lambda]+[\Omega,\Gamma] && (16)\\
[\Omega\Lambda,\Gamma] &= \Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda && (17)
\end{aligned}$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 14 we could also have written

$$\{\omega,\lambda\sigma\}=\sigma\{\omega,\lambda\}+\lambda\{\omega,\sigma\} \tag{18}$$

since all three quantities are numerical (not operators), so multiplication commutes. In 17 it is *not* true in general that, for example

$$\Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda=[\Lambda,\Gamma]\Omega+[\Omega,\Gamma]\Lambda \tag{19}$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$\{\omega(x,p),\lambda(x,p)\}=\gamma(x,p) \tag{20}$$

by

$$[\Omega(X,P),\Lambda(X,P)] = i\hbar\Gamma(X,P) \tag{21}$$

where each of the operators in the last equation is obtained by replacing $x$ in the first equation by $X$ and $p$ by $P$. We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

$$\begin{aligned}\Omega &= X &(22)\\ \Lambda &= X^2+P^2 &(23)\end{aligned}$$

In the classical version, we calculate the Poisson bracket

$$\begin{aligned}\{\omega,\lambda\} &= \left\{x,x^2+p^2\right\} &(24)\\ &= \left\{x,x^2\right\}+\left\{x,p^2\right\} &(25)\\ &= 0+2\{x,p\}p &(26)\\ &= 2p &(27)\end{aligned}$$

Thus, by our rule above, the quantum version should be

$$[\Omega,\Lambda] = 2i\hbar P \tag{28}$$

We can verify this using 17

$$\begin{aligned}\left[X,X^2+P^2\right] &= \left[X,X^2\right]+\left[X,P^2\right] &(29)\\ &= 0-\left[P^2,X\right] &(30)\\ &= -P[P,X]-[P,X]P &(31)\\ &= -P(-i\hbar)-(-i\hbar)P &(32)\\ &= 2i\hbar P &(33)\end{aligned}$$

In this case, there is no ordering ambiguity in the quantum version, since $[X,P] = i\hbar$ is just a number.

For a second example, suppose we have

$$\begin{aligned}\Omega &= X^2 &(34)\\ \Lambda &= P^2 &(35)\end{aligned}$$

The classical version gives us, using the relations 14, 11 and 27

$$\begin{aligned}
\left\{x^2,p^2\right\} &= -\left\{p^2,x^2\right\} && (36)\\
&= -2\left\{p^2,x\right\}x && (37)\\
&= 2\left\{x,p^2\right\}x && (38)\\
&= 4px && (39)
\end{aligned}$$

In the classical case, this result is the same as $4xp$, but because $X$ and $P$ don't commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

$$\left[X^2,P^2\right] = X\left[X,P^2\right] + \left[X,P^2\right]X \tag{40}$$

From 33 we have

$$\left[X,P^2\right] = 2i\hbar P \tag{41}$$

so we get

$$\left[X^2,P^2\right] = 2i\hbar\left(XP+PX\right) \tag{42}$$

Thus if the Poisson bracket involves a product of $p$ and $x$, this should be replaced by

$$xp \text{ or } px \to \frac{1}{2}\left(XP+PX\right) \tag{43}$$

in the quantum version.

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