

POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlier included specifications for the matrix elements of position X and momentum P in position space:

$$\begin{aligned}(1) \quad \langle x|X|x' \rangle &= x\delta(x-x') \\(2) \quad \langle x|P|x' \rangle &= -i\hbar\delta'(x-x')\end{aligned}$$

A more fundamental form of this postulate is to specify the commutation relation between X and P , which is independent of the basis and is

$$(3) \quad [X, P] = i\hbar$$

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$(4) \quad X = i\hbar \frac{d}{dp}$$

$$(5) \quad P = p$$

We can verify this by calculating the commutator by applying it to a function $f(p)$:

$$(6) \quad [X, P]f = i\hbar \frac{d}{dp}(pf(p)) - i\hbar p \frac{d}{dp}f(p)$$

$$(7) \quad = i\hbar f(p) + i\hbar p \frac{d}{dp}f(p) - i\hbar p \frac{d}{dp}f(p)$$

$$(8) \quad = i\hbar f(p)$$

Thus 3 is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracket for two physical quantities in the classical system, which gives

$$(9) \quad \{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right)$$

where q_i and p_i are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by $i\hbar$ times the corresponding commutator. That is

$$(10) \quad [\Omega, \Lambda] = i\hbar \{\omega, \lambda\}$$

For the case of X and P , we have, in classical mechanics in one dimension

$$(11) \quad \{x, p\} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} = 1$$

so the quantum commutator is given by 3.

For other quantities, we can use the theorems on the Poisson bracketsto reduce them:

$$(12) \quad \{\omega, \lambda\} = -\{\lambda, \omega\}$$

$$(13) \quad \{\omega, \lambda + \sigma\} = \{\omega, \lambda\} + \{\omega, \sigma\}$$

$$(14) \quad \{\omega, \lambda \sigma\} = \{\omega, \lambda\} \sigma + \{\omega, \sigma\} \lambda$$

Quantum commutators obey similar rules

$$(15) \quad [\Omega, \Lambda] = -[\Lambda, \Omega]$$

$$(16) \quad [\Omega, \Lambda + \Gamma] = [\Omega, \Lambda] + [\Omega, \Gamma]$$

$$(17) \quad [\Omega \Lambda, \Gamma] = \Omega [\Lambda, \Gamma] + [\Omega, \Gamma] \Lambda$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 14 we could also have written

$$(18) \quad \{\omega, \lambda \sigma\} = \sigma \{\omega, \lambda\} + \lambda \{\omega, \sigma\}$$

since all three quantities are numerical (not operators), so multiplication commutes. In 17 it is *not* true in general that, for example

$$(19) \quad \Omega [\Lambda, \Gamma] + [\Omega, \Gamma] \Lambda = [\Lambda, \Gamma] \Omega + [\Omega, \Gamma] \Lambda$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$(20) \quad \{\omega(x, p), \lambda(x, p)\} = \gamma(x, p)$$

by

$$(21) \quad [\Omega(X, P), \Lambda(X, P)] = i\hbar\Gamma(X, P)$$

where each of the operators in the last equation is obtained by replacing x in the first equation by X and p by P . We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

$$(22) \quad \Omega = X$$

$$(23) \quad \Lambda = X^2 + P^2$$

In the classical version, we calculate the Poisson bracket

$$(24) \quad \{\omega, \lambda\} = \{x, x^2 + p^2\}$$

$$(25) \quad = \{x, x^2\} + \{x, p^2\}$$

$$(26) \quad = 0 + 2\{x, p\}p$$

$$(27) \quad = 2p$$

Thus, by our rule above, the quantum version should be

$$(28) \quad [\Omega, \Lambda] = 2i\hbar P$$

We can verify this using 17

$$(29) \quad [X, X^2 + P^2] = [X, X^2] + [X, P^2]$$

$$(30) \quad = 0 - [P^2, X]$$

$$(31) \quad = -P[P, X] - [P, X]P$$

$$(32) \quad = -P(-i\hbar) - (-i\hbar)P$$

$$(33) \quad = 2i\hbar P$$

In this case, there is no ordering ambiguity in the quantum version, since $[X, P] = i\hbar$ is just a number.

For a second example, suppose we have

$$(34) \quad \Omega = X^2$$

$$(35) \quad \Lambda = P^2$$

The classical version gives us, using the relations 14, 11 and 27

$$\begin{aligned}
 (36) \quad \{x^2, p^2\} &= -\{p^2, x^2\} \\
 (37) \quad &= -2\{p^2, x\}x \\
 (38) \quad &= 2\{x, p^2\}x \\
 (39) \quad &= 4px
 \end{aligned}$$

In the classical case, this result is the same as $4xp$, but because X and P don't commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

$$(40) \quad [X^2, P^2] = X [X, P^2] + [X, P^2] X$$

From 33 we have

$$(41) \quad [X, P^2] = 2i\hbar P$$

so we get

$$(42) \quad [X^2, P^2] = 2i\hbar (XP + PX)$$

Thus if the Poisson bracket involves a product of p and x , this should be replaced by

$$(43) \quad xp \text{ or } px \rightarrow \frac{1}{2}(XP + PX)$$

in the quantum version.

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