POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlierincluded specifications for the matrix elements of position X and momentum P in position space:

$$\left\langle x\left|X\right|x'\right\rangle = x\delta\left(x-x'\right) \tag{1}$$

$$\left\langle x\left|P\right|x'\right\rangle = -i\hbar\delta'\left(x-x'\right) \tag{2}$$

A more fundamental form of this postulate is to specify the commutation relation between X and P, which is independent of the basis and is

$$[X,P]=i\hbar \tag{3}$$

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$X = i\hbar\frac{d}{dp} \tag{4}$$

$$P = p \tag{5}$$

We can verify this by calculating the commutator by applying it to a function $f(p)$:

$$[X,P]f = i\hbar\frac{d}{dp}\left(pf(p)\right)-i\hbar p\frac{d}{dp}f(p) \tag{6}$$

$$= i\hbar f(p)+i\hbar p\frac{d}{dp}f(p)-i\hbar p\frac{d}{dp}f(p) \tag{7}$$

$$= i\hbar f(p) \tag{8}$$

Thus 3 is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracketfor two physical quantities in the classical system, which gives

$$\{\omega,\lambda\}=\sum_i\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i}-\frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right) \tag{9}$$

where q_i and p_i are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by $i\hbar$ times the corresponding commutator. That is

$$[\Omega,\Lambda]=i\hbar\{\omega,\lambda\} \tag{10}$$

For the case of X and P, we have, in classical mechanics in one dimension

$$\{x,p\}=\frac{\partial x}{\partial x}\frac{\partial p}{\partial p}-\frac{\partial x}{\partial p}\frac{\partial p}{\partial x}=1 \tag{11}$$

so the quantum commutator is given by 3.

For other quantities, we can use the theorems on the Poisson bracketsto reduce them:

$$\{\omega,\lambda\} = -\{\lambda,\omega\} \tag{12}$$
$$\{\omega,\lambda+\sigma\} = \{\omega,\lambda\}+\{\omega,\sigma\} \tag{13}$$
$$\{\omega,\lambda\sigma\} = \{\omega,\lambda\}\sigma+\{\omega,\sigma\}\lambda \tag{14}$$

Quantum commutators obey similar rules

$$[\Omega,\Lambda] = -[\Lambda,\Omega] \tag{15}$$
$$[\Omega,\Lambda+\Gamma] = [\Omega,\Lambda]+[\Omega,\Gamma] \tag{16}$$
$$[\Omega\Lambda,\Gamma] = \Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda \tag{17}$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 14 we could also have written

$$\{\omega,\lambda\sigma\}=\sigma\{\omega,\lambda\}+\lambda\{\omega,\sigma\} \tag{18}$$

since all three quantities are numerical (not operators), so multiplication commutes. In 17 it is *not* true in general that, for example

$$\Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda=[\Lambda,\Gamma]\Omega+[\Omega,\Gamma]\Lambda \tag{19}$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$\{\omega(x,p),\lambda(x,p)\}=\gamma(x,p) \tag{20}$$

by

$$[\Omega(X,P),\Lambda(X,P)]=i\hbar\Gamma(X,P) \tag{21}$$

where each of the operators in the last equation is obtained by replacing x in the first equation by X and p by P. We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

$$\begin{aligned}\Omega &= X && (22)\\ \Lambda &= X^2+P^2 && (23)\end{aligned}$$

In the classical version, we calculate the Poisson bracket

$$\begin{aligned}\{\omega,\lambda\} &= \left\{x,x^2+p^2\right\} && (24)\\ &= \left\{x,x^2\right\}+\left\{x,p^2\right\} && (25)\\ &= 0+2\{x,p\}p && (26)\\ &= 2p && (27)\end{aligned}$$

Thus, by our rule above, the quantum version should be

$$[\Omega,\Lambda]=2i\hbar P \tag{28}$$

We can verify this using 17

$$\begin{aligned}\left[X,X^2+P^2\right] &= \left[X,X^2\right]+\left[X,P^2\right] && (29)\\ &= 0-\left[P^2,X\right] && (30)\\ &= -P[P,X]-[P,X]P && (31)\\ &= -P(-i\hbar)-(-i\hbar)P && (32)\\ &= 2i\hbar P && (33)\end{aligned}$$

In this case, there is no ordering ambiguity in the quantum version, since $[X,P]=i\hbar$ is just a number.

For a second example, suppose we have

$$\begin{aligned}\Omega &= X^2 && (34)\\ \Lambda &= P^2 && (35)\end{aligned}$$

The classical version gives us, using the relations 14, 11 and 27

$$\left\{x^{2},p^{2}\right\} = -\left\{p^{2},x^{2}\right\} \tag{36}$$
$$= -2\left\{p^{2},x\right\}x \tag{37}$$
$$= 2\left\{x,p^{2}\right\}x \tag{38}$$
$$= 4px \tag{39}$$

In the classical case, this result is the same as $4xp$, but because X and P don't commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

$$\left[X^{2},P^{2}\right] = X\left[X,P^{2}\right]+\left[X,P^{2}\right]X \tag{40}$$

From 33 we have

$$\left[X,P^{2}\right] = 2i\hbar P \tag{41}$$

so we get

$$\left[X^{2},P^{2}\right] = 2i\hbar\left(XP+PX\right) \tag{42}$$

Thus if the Poisson bracket involves a product of p and x, this should be replaced by

$$xp \text{ or } px \rightarrow \frac{1}{2}\left(XP+PX\right) \tag{43}$$

in the quantum version.

PINGBACKS

Pingback: Angular momentum - Poisson bracket to commutator

Pingback: Hamiltonian in non-rectangular coordinates

Pingback: Harmonic oscillator: momentum space functions and Hermite polynomial recursion

Pingback: Direct product of two vector spaces