ANGULAR MOMENTUM - POISSON BRACKET TO COMMUTATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.8.

The classical angular momentum components are

$$\ell_x = yp_z - zp_y \tag{1}$$

$$\ell_y = zp_x - xp_z \tag{2}$$

$$\ell_z = xp_y - yp_x \tag{3}$$

In the position basis, we can replace each coordinate by its quantum operator $x \to X$, $y \to Y$ and $z \to Z$, and each momentum component by the derivative $p_i \to -i\hbar\partial/\partial q_i$, where q_i is the ith coordinate. This gives

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{4}$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{5}$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{6}$$

Because coordinates always commute with momentum components of other coordinates (x commutes with p_y and p_z, etc), there is no ordering ambiguity in making the transition from classical to quantum mechanics. That is, we could place the coordinate on either side of the momentum in each term for all components L_i.

Classically, we can calculate the Poisson brackets for the angular momentum components. For example

$$\left\{\ell_x, \ell_y\right\} = \sum_i \left(\frac{\partial \ell_x}{\partial q_i}\frac{\partial \ell_y}{\partial p_i} - \frac{\partial \ell_x}{\partial p_i}\frac{\partial \ell_y}{\partial q_i}\right) \tag{7}$$

$$= -p_y(-x) - yp_z \tag{8}$$

$$= xp_y - yp_z \tag{9}$$

$$= \ell_z \tag{10}$$

According to the rule for converting classical Poisson brackets to quantum commutators, we should get (since there is no ordering ambiguity)

$$[L_x, L_y] = i\hbar L_z \tag{11}$$

As we've seen earlier, this is verified by direct calculation using the position-momentum commutator

$$\left[q_i, p_j\right] = i\hbar\delta_{ij} \tag{12}$$