

CHANGING THE POSITION BASIS WITH A UNITARY TRANSFORMATION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.9.

The standard representation of the position and momentum operators in the position basis is

$$X \rightarrow x \quad (1)$$

$$P \rightarrow -i\hbar \frac{d}{dx} \quad (2)$$

It turns out it's possible to modify this definition by adding some arbitrary function of position $f(x)$ to P so we have

$$X' \rightarrow x \quad (3)$$

$$P' \rightarrow -i\hbar \frac{d}{dx} + f(x) \quad (4)$$

Since any function of x commutes with X , the commutation relations remain unchanged, so we have

$$[X', P'] = i\hbar \quad (5)$$

Another way of interpreting this change in operators is by using the unitary transformation of the X basis, in the form

$$|x\rangle \rightarrow |\tilde{x}\rangle = e^{ig(X)/\hbar} |x\rangle = e^{ig(x)/\hbar} |x\rangle \quad (6)$$

where

$$g(x) \equiv \int^x f(x') dx' \quad (7)$$

The last equality in 6 comes from the fact that operating on $|x\rangle$ with any function of the X operator (provided the function can be expanded in a power series) results in multiplying $|x\rangle$ by the same function, but with the operator X replaced by the numeric position value.

To verify this works, we can calculate the matrix elements of the old X and P operators in the new basis. We have

$$\langle \tilde{x}|X|\tilde{x}'\rangle = \left\langle x \left| e^{-ig(x)/\hbar} X e^{ig(x')/\hbar} \right| x' \right\rangle \quad (8)$$

At this stage, since the two exponentials are numerical functions and not operators, we can take them outside the bracket to

$$\langle \tilde{x}|X|\tilde{x}'\rangle = e^{-ig(x)/\hbar} e^{ig(x')/\hbar} \langle x|X|x'\rangle \quad (9)$$

$$= e^{-ig(x)/\hbar} e^{ig(x')/\hbar} x' \delta(x-x') \quad (10)$$

$$= x \delta(x-x') \quad (11)$$

The exponentials cancel in the last line since the delta function is non-zero only when $x = x'$.

The above result can also be obtained by inserting a couple of identity operators into 8:

$$\langle x \left| e^{-ig(x)/\hbar} X e^{ig(x')/\hbar} \right| x' \rangle = \int \int \langle x \left| e^{-ig(x)/\hbar} \right| y \rangle \langle y|X|z\rangle \langle z \left| e^{ig(x')/\hbar} \right| x' \rangle dy dz \quad (12)$$

$$= \int \int \langle x \left| e^{-ig(x)/\hbar} \right| y \rangle z \delta(y-z) \langle z \left| e^{ig(x')/\hbar} \right| x' \rangle dy dz \quad (13)$$

$$= \int \langle x \left| e^{-ig(x)/\hbar} \right| z \rangle z \langle z \left| e^{ig(x')/\hbar} \right| x' \rangle dz \quad (14)$$

$$= \int e^{i[g(x')-g(x)]/\hbar} \langle x|z\rangle z \langle z|x'\rangle dz \quad (15)$$

$$= \int e^{i[g(x')-g(x)]/\hbar} \delta(x-z) z \delta(z-x') dz \quad (16)$$

$$= e^{i[g(x')-g(x)]/\hbar} x' \delta(x-x') \quad (17)$$

$$= x \delta(x-x') \quad (18)$$

The momentum operator works as follows. Using the original definition 2 on the modified basis we have

$$\langle \tilde{x}|P|\tilde{x}'\rangle = -i\hbar \left\langle x \left| e^{-ig(x)/\hbar} \frac{d}{dx'} e^{ig(x')/\hbar} \right| x' \right\rangle \quad (19)$$

$$= -i\hbar \left\langle x \left| e^{-ig(x)/\hbar} \frac{i}{\hbar} e^{ig(x')/\hbar} \frac{dg(x')}{dx'} \right| x' \right\rangle - \quad (20)$$

$$i\hbar \left\langle x \left| e^{-ig(x)/\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \right| x' \right\rangle \quad (21)$$

From 7 we have

$$\frac{dg(x)}{dx} = \frac{d}{dx} \int^x f(x') dx' = f(x) \quad (22)$$

This gives

$$\langle \tilde{x} | P | \tilde{x}' \rangle = \left\langle x \left| e^{i[g(x')-g(x)]/\hbar} \left[f(x') - i\hbar \frac{d}{dx'} \right] \right| x' \right\rangle \quad (23)$$

$$= e^{i[g(x')-g(x)]/\hbar} \left[f(x') - i\hbar \frac{d}{dx'} \right] \langle x | x' \rangle \quad (24)$$

$$= e^{i[g(x')-g(x)]/\hbar} \left[f(x') - i\hbar \frac{d}{dx'} \right] \delta(x-x') \quad (25)$$

$$= \left[f(x) - i\hbar \frac{d}{dx} \right] \delta(x-x') \quad (26)$$

This shows that by a unitary change of X basis 6, we transform the position and momentum operators (well, just the momentum operator, really) according to 3. We've multiplied the original $|x\rangle$ states by a phase factor which depends on some function $f(x)$. This doesn't change the matrix elements of X , but it does add $f(x)$ to the matrix elements of P . The commonly used definition of P is thus with $f(x) = 0$.