CHANGING THE POSITION BASIS WITH A UNITARY TRANSFORMATION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.9.

The standard representation of the position and momentum operators in the position basis is

$$X \rightarrow x \tag{1}$$

$$P \rightarrow -i\hbar\frac{d}{dx} \tag{2}$$

It turns out it's possible to modify this definition by adding some arbitrary function of position $f(x)$ to P so we have

$$X' \rightarrow x \tag{3}$$

$$P' \rightarrow -i\hbar\frac{d}{dx}+f(x) \tag{4}$$

Since any function of x commutes with X, the commutation relations remain unchanged, so we have

$$\left[X',P'\right]=i\hbar \tag{5}$$

Another way of interpreting this change in operators is by using the unitary transformation of the X basis, in the form

$$|x\rangle \rightarrow |\tilde{x}\rangle = e^{ig(X)/\hbar}|x\rangle = e^{ig(x)/\hbar}|x\rangle \tag{6}$$

where

$$g(x) \equiv \int^{x} f\left(x'\right)dx' \tag{7}$$

The last equality in 6 comes from the fact that operating on $|x\rangle$ with any function of the X operator (provided the function can be expanded in a power series) results in multiplying $|x\rangle$ by the same function, but with the operator X replaced by the numeric position value.

To verify this works, we can calcuate the matrix elements of the old X and P operators in the new basis. We have

$$\left\langle \tilde{x}\left|X\right|\tilde{x}'\right\rangle =\left\langle x\left|e^{-ig(x)/\hbar}Xe^{ig(x')/\hbar}\right|x'\right\rangle \tag{8}$$

At this stage, since the two exponentials are numerical functions and not operators, we can take them outside the bracket to

$$\begin{aligned}
\left\langle \tilde{x}\left|X\right|\tilde{x}'\right\rangle &= e^{-ig(x)/\hbar}e^{ig(x')/\hbar}\left\langle x\left|X\right|x'\right\rangle \qquad (9)\\
&= e^{-ig(x)/\hbar}e^{ig(x')/\hbar}x'\delta\left(x-x'\right) \qquad (10)\\
&= x\delta\left(x-x'\right) \qquad (11)
\end{aligned}$$

The exponentials cancel in the last line since the delta function is non-zero only when $x=x'$.

The above result can also be obtained by inserting a couple of identity operators into 8:

$$\begin{aligned}
\left\langle x\left|e^{-ig(x)/\hbar}Xe^{ig(x')/\hbar}\right|x'\right\rangle &= \int\int\left\langle x\left|e^{-ig(x)/\hbar}\right|y\right\rangle \left\langle y\left|X\right|z\right\rangle \left\langle z\left|e^{ig(x')/\hbar}\right|x'\right\rangle dy\,dz \qquad (12)\\
&= \int\int\left\langle x\left|e^{-ig(x)/\hbar}\right|y\right\rangle z\delta\left(y-z\right)\left\langle z\left|e^{ig(x')/\hbar}\right|x'\right\rangle dy\,dz \qquad (13)\\
&= \int\left\langle x\left|e^{-ig(x)/\hbar}\right|z\right\rangle z\left\langle z\left|e^{ig(x')/\hbar}\right|x'\right\rangle dz \qquad (14)\\
&= \int e^{i[g(x')-g(x)]/\hbar}\left\langle x\left|z\right.\right\rangle z\left\langle z\left|x'\right.\right\rangle dz \qquad (15)\\
&= \int e^{i[g(x')-g(x)]/\hbar}\delta\left(x-z\right)z\delta\left(z-x'\right)dz \qquad (16)\\
&= e^{i[g(x')-g(x)]/\hbar}x'\delta\left(x-x'\right) \qquad (17)\\
&= x\delta\left(x-x'\right) \qquad (18)
\end{aligned}$$

The momentum operator works as follows. Using the original definition 2 on the modified basis we have

$$\left\langle \tilde{x}\left|P\right|\tilde{x}'\right\rangle = -i\hbar\left\langle x\left|e^{-ig(x)/\hbar}\frac{d}{dx'}e^{ig(x')/\hbar}\right|x'\right\rangle \tag{19}$$

$$= -i\hbar\left\langle x\left|e^{-ig(x)/\hbar}\frac{i}{\hbar}e^{ig(x')/\hbar}\frac{dg(x')}{dx'}\right|x'\right\rangle - \tag{20}$$

$$i\hbar\left\langle x\left|e^{-ig(x)/\hbar}e^{ig(x')/\hbar}\frac{d}{dx'}\right|x'\right\rangle \tag{21}$$

From 7 we have

$$\frac{dg(x)}{dx} = \frac{d}{dx}\int^{x} f\left(x'\right)dx' = f(x) \tag{22}$$

This gives

$$\left\langle \tilde{x}\left|P\right|\tilde{x}'\right\rangle = \left\langle x\left|e^{i[g(x')-g(x)]/\hbar}\left[f\left(x'\right) - i\hbar\frac{d}{dx'}\right]\right|x'\right\rangle \tag{23}$$

$$= e^{i[g(x')-g(x)]/\hbar}\left[f\left(x'\right) - i\hbar\frac{d}{dx'}\right]\left\langle x\left|x'\right.\right\rangle \tag{24}$$

$$= e^{i[g(x')-g(x)]/\hbar}\left[f\left(x'\right) - i\hbar\frac{d}{dx'}\right]\delta\left(x-x'\right) \tag{25}$$

$$= \left[f(x) - i\hbar\frac{d}{dx}\right]\delta\left(x-x'\right) \tag{26}$$

This shows that by a unitary change of X basis 6, we transform the position and momentum operators (well, just the momentum operator, really) according to 3. We've multiplied the original $|x\rangle$ states by a phase factor which depends on some function $f(x)$. This doesn't change the matrix elements of X, but it does add $f(x)$ to the matrix elements of P. The commonly used definition of P is thus with $f(x) = 0$.