HARMONIC OSCILLATOR: MOMENTUM SPACE FUNCTIONS AND HERMITE POLYNOMIAL RECURSION RELATIONS FROM RAISING AND LOWERING OPERATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.5, Exercises 7.5.1 - 7.5.3.

Earlier, we found the position space energy eigenfunctions of the harmonic oscillator to be

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2} \tag{1}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar} \tag{2}$$

where y in the first equation is shorthand for

$$y = \sqrt{\frac{m\omega}{\hbar}}x \tag{3}$$

It turns out that an alternative method for deriving these functions uses the lowering operator a. Shankar gives the derivation of $\psi_n(x)$ in his section 7.5, but we can use the same technique to derive the momentum space functions. We start with the ground state and use

$$a|0\rangle = 0 \tag{4}$$

In terms of X and P, we have

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2m\omega\hbar}}P \tag{5}$$

To find the momentum space functions, we need to express X and P in terms of p:

$$X = i\hbar\frac{d}{dp} \tag{6}$$

$$P = p \tag{7}$$

We thus have

$$\left[i\hbar\sqrt{\frac{m\omega}{2\hbar}}\frac{d}{dp}+i\frac{1}{\sqrt{2m\omega\hbar}}p\right]\psi_0(p)=0 \tag{8}$$

If we define the auxiliary variable

$$z\equiv\frac{p}{\sqrt{\hbar m\omega}} \tag{9}$$

we get

$$\left(\frac{d}{dz}+z\right)\psi_0(z)=0 \tag{10}$$

This has the solution

$$\psi_0(z)=Ae^{-z^2/2} \tag{11}$$

for some normalization constant A. Thus in terms of p we have

$$\psi_0(p)=Ae^{-p^2/2\hbar m\omega} \tag{12}$$

Normalizing in the usual way, making use of the Gaussian integral, we have

$$\int_{-\infty}^{\infty}\psi_0^2(p)\,dp = A^2\int_{-\infty}^{\infty}e^{-p^2/\hbar m\omega}dp=1 \tag{13}$$

$$A = \frac{1}{(\pi\hbar m\omega)^{1/4}} \tag{14}$$

This agrees with the earlier result which was obtained by solving a second-order differential equation.

We can also use a and $a^\dagger$ to verify a couple of recursion relations for Hermite polynomials. Reverting back to position space we have

$$X = x \tag{15}$$

$$P = -i\hbar\frac{d}{dx} \tag{16}$$

so 5 becomes

$$a=\sqrt{\frac{m\omega}{2\hbar}}x+\frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx} \tag{17}$$

Also from 5 we have, since X and P are both hermitian operators

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2m\omega\hbar}}P \tag{18}$$

$$= \sqrt{\frac{m\omega}{2\hbar}}x - \frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx} \tag{19}$$

Defining

$$y \equiv \sqrt{\frac{m\omega}{\hbar}}x \tag{20}$$

we have

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) \tag{21}$$

$$a^{\dagger} = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) \tag{22}$$

We also recall the normalization conditions on the raising and lowering operators:

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{23}$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{24}$$

Applying 23 to 1 we have, after cancelling common factors from each side:

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(y + \frac{d}{dy}\right)\left[H_n(y)e^{-y^2/2}\right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)e^{-y^2/2} \tag{25}$$

$$\frac{1}{2\sqrt{n}}\frac{1}{\sqrt{2^{n-1}(n-1)!}}e^{-y^2/2}\left[yH_n(y) - yH_n(y) + \frac{dH_n}{dy}\right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)e^{-y^2/2} \tag{26}$$

$$yH_n(y) - yH_n(y) + \frac{dH_n}{dy} = 2nH_{n-1}(y) \tag{27}$$

$$H_n'(y) = 2nH_{n-1}(y) \tag{28}$$

Another recursion relation for Hermite polynomials can be found as follows. We start with 22 to get

$$a+a^{\dagger}=\sqrt{2}y \tag{29}$$

We now apply 23 and 24 to 1. We can cancel common factors, including $e^{-y^2/2}$, from both sides to get

$$\left(a+a^{\dagger}\right)\psi_n = \sqrt{2}y\psi_n \tag{30}$$

$$\frac{\sqrt{2}y}{\sqrt{2^n n!}}H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)+\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)!}}H_{n+1}(y) \tag{31}$$

$$\frac{y}{\sqrt{2^{n-1}n(n-1)!}}H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)+\frac{1}{2\sqrt{2^{n-1}n(n-1)!}}H_{n+1}(y) \tag{32}$$

$$yH_n(y) = nH_{n-1}(y)+\frac{1}{2}H_{n+1}(y) \tag{33}$$

$$H_{n+1}(y) = 2yH_n(y)-2nH_{n-1}(y) \tag{34}$$