PATH INTEGRALS FOR SPECIAL POTENTIALS; USE OF CLASSICAL ACTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 8. Section 8.6, Exercises 8.6.1 - 8.6.2.

We've seen that if we use the path integral formulation for a free particle, we get the exact propagator by considering only one path (the classical path) between the starting point (x',t') and the end point (x,t). In this case, the propagator has the form

$$U\left(x,t;x',t'\right)=A\left(t\right)e^{iS_{cl}/\hbar} \tag{1}$$

where S_{cl} is the classical action. It turns out that this form is true for a wider set of potentials, beyond just the free particle. The general form of the potential for which this is true is

$$V=a+bx+cx^{2}+d\dot{x}+ex\dot{x} \tag{2}$$

where a,b,c,d and e are constants. The general expression for the propagator is (where we're taking the starting time to be $t'=0$):

$$U\left(x,t;x'\right)=\int_{x'}^{x}e^{iS[x(t'')]/\hbar}\mathfrak{D}\left[x\left(t''\right)\right] \tag{3}$$

where the notation $\mathfrak{D}\left[x\left(t''\right)\right]$ means an integration over all possible paths from x' to x in the given time interval.

For a given path, we can write the location of the particle $x\left(t''\right)$ as composed of its position on the classical path $x_{cl}\left(t''\right)$ plus the deviation $y\left(t''\right)$ from the classical path:

$$x\left(t''\right)=x_{cl}\left(t''\right)+y\left(t''\right) \tag{4}$$

As the endpoints are fixed

$$y\left(0\right)=y\left(t\right)=0 \tag{5}$$

Also, since for any given potential and choice of endpoints, $x_{cl}\left(t''\right)$ is fixed for all times, it is effectively a constant with regard to the path integration. Therefore

$$dx=dy \tag{6}$$

Making these substitutions into 3, we get, using Shankar's slightly misleading notation:

$$U\left(x,t;x'\right)=\int_0^0 e^{iS[x_{cl}(t'')+y(t'')]/\hbar}\mathfrak{D}\left[y\left(t''\right)\right] \tag{7}$$

Usually, when the limits on an integral are the same, the integral evaluates to zero. However, in this case, the notation $\int_0^0 \mathfrak{D}\left[y\left(t''\right)\right]$ means that y starts and ends at zero, but covers all possible paths between these endpoints.

The action is the integral of the Lagrangian which, for the potential 2 is

$$\begin{aligned} L &= T-V \qquad (8)\\ &= \frac{1}{2}m\dot{x}^2-a-bx-cx^2-d\dot{x}-ex\dot{x} \qquad (9)\end{aligned}$$

Because L is quadratic in both x and $\dot{x}$, we can expand it in a Taylor series up to second order without any approximation. That is

$$L\left(x_{cl}+y,\dot{x}_{cl}+\dot{y}\right)=L\left(x_{cl},\dot{x}_{cl}\right)+\left.\frac{\partial L}{\partial x}\right|_{x_{cl}}y+\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\dot{y}+ \tag{10}$$

$$\frac{1}{2}\left(\left.\frac{\partial^2 L}{\partial x^2}\right|_{x_{cl}}y^2+2\left.\frac{\partial^2 L}{\partial x\partial \dot{x}}\right|_{x_{cl}}y\dot{y}+\left.\frac{\partial^2 L}{\partial \dot{x}^2}\right|_{x_{cl}}\dot{y}^2\right) \tag{11}$$

Look first at the last two terms on the RHS of the first line. Using the equations of motion, we have

$$\left.\frac{\partial L}{\partial x}\right|_{x_{cl}}=\frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\right) \tag{12}$$

To get the action, we need to integrate the Lagrangian over the time interval of interest. Integrating these two terms gives

$$\int_0^t\left[\left.\frac{\partial L}{\partial x}\right|_{x_{cl}}y+\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\dot{y}\right]dt''=\int_0^t\left[\frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\right)y+\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\dot{y}\right]dt'' \tag{13}$$

$$=\left(\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\right)y\Bigg|_0^t-\int_0^t\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\dot{y}dt''+\int_0^t\left.\frac{\partial L}{\partial \dot{x}}\right|_{x_{cl}}\dot{y}dt'' \tag{14}$$

$$=0 \tag{15}$$

where we integrated the first term by parts. The integrated term in the second line is zero because $y=0$ at both endpoints, and the last two terms cancel each other.

Returning to 11, we can calculate the three second derivatives explicitly:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 L}{\partial x^2} &= -c && (16)\\
\left.\frac{\partial^2 L}{\partial x \partial \dot{x}}\right|_{x_{cl}} &= -e && (17)\\
\frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^2} &= \frac{m}{2} && (18)
\end{aligned}$$

[Note that Shankar's equation 8.6.10 is wrong - the RHS should be $\frac{m}{2}$. However, his equation 8.6.11 appears to be correct. Thanks to commenter Alex for pointing this out.]

The integral of the first term on the RHS of 10 is just the classical action, so we get for the propagator 7:

$$U\left(x,t;x'\right) = e^{iS_{cl}/\hbar}\int_0^0 \exp\left[\frac{i}{\hbar}\int_0^t\left(\frac{m\dot{y}^2}{2} - cy^2 - ey\dot{y}\right)dt''\right]\mathfrak{D}\left[y\left(t''\right)\right] \tag{19}$$

The remaining path integral can still be difficult to evaluate, but we can observe a few properties that it has. First, for any given path in the path integral, we must be able to express both y and $\dot{y}$ as functions of time t'', so the complete path integral can depend only on the end time t (and, of course, on the constants m, c and e). That is, the propagator will always have the form 1:

$$U\left(x,t;x',t'\right) = A\left(t\right)e^{iS_{cl}/\hbar} \tag{20}$$

We have already evaluated the integral for the free particle where $c=e=0$ and we found there that

$$U\left(x,t;x'\right) = \sqrt{\frac{m}{2\pi\hbar it}}e^{iS_{cl}/\hbar} \tag{21}$$

Since the constant b doesn't appear in 19, the propagator must have the same form for the more general case where $V=a+bx$. For more complex potentials, such as the harmonic oscillator, the function $A\left(t\right)$ will in general have a different form and will have to be calculated explicitly in these cases.

As an example, we'll consider the case of a particle subject to a constant force in the x direction, so that the potential is given by

$$V(x) = -fx \tag{22}$$

This gives a constant force of

$$F = -\frac{dV}{dx} = f \tag{23}$$

and thus a constant acceleration of f/m. For such a particle, its classical position is (from first year physics)

$$x_{cl}(t'') = x_0 + v_0 t'' + \frac{1}{2}\frac{f}{m}t''^2 \tag{24}$$

$$\dot{x}_{cl}(t'') = v_0 + \frac{f}{m}t'' \tag{25}$$

To find x_0 and v_0, we impose boundary conditions. At $t'' = 0$

$$x_{cl}(0) = x_0 = x' \tag{26}$$

At $t'' = t$, its position is

$$x_{cl}(t) = x = x' + v_0 t + \frac{f}{2m}t^2 \tag{27}$$

This gives

$$v_0 = \frac{x - x'}{t} - \frac{f}{2m}t \tag{28}$$

The classical Lagrangian is

$$L = T - V \tag{29}$$

$$= \frac{1}{2}m\dot{x}_{cl}^2 + fx_{cl} \tag{30}$$

$$= \frac{1}{2}m\left(v_0 + \frac{f}{m}t''\right)^2 + f\left(x_0 + v_0 t'' + \frac{1}{2}\frac{f}{m}t''^2\right) \tag{31}$$

$$= \frac{1}{2}m\left(\frac{x-x'}{t} - \frac{f}{2m}t + \frac{f}{m}t''\right)^2 + f\left(x' + \left(\frac{x-x'}{t} - \frac{f}{2m}t\right)t'' + \frac{1}{2}\frac{f}{m}t''^2\right) \tag{32}$$

Note that t is a constant, as it is the time of the endpoint of the motion. To find the classical action, we must integrate this from $t'' = 0$ to t. The integral is a straightforward integral of a quadratic in t'', although the algebra is tedious if done by hand, so is best done with Maple.

$$S_{cl} = \int_0^t L\, dt'' \tag{33}$$

$$= \frac{1}{3}\frac{f^2t^3}{m} + \left(\frac{x-x'}{t} - \frac{1}{2}\frac{ft}{m}\right) ft^2 + \frac{1}{2}m\left(\frac{x-x'}{t} - \frac{1}{2}\frac{ft}{m}\right)^2 t + fxt \tag{34}$$

$$= -\frac{f^2t^3}{24m} + \frac{1}{2}ft\left(x+x'\right) + \frac{m\left(x-x'\right)^2}{2t} \tag{35}$$

From 21, this gives a propagator of

$$U\left(x,t;x'\right) = \sqrt{\frac{m}{2\pi\hbar it}}\exp\left[\frac{i}{\hbar}\left(-\frac{f^2t^3}{24m} + \frac{1}{2}ft\left(x+x'\right) + \frac{m\left(x-x'\right)^2}{2t}\right)\right] \tag{36}$$

This agrees with Shankar's result in his equation 5.4.31.

As another example, consider the harmonic oscillator, where the potential is

$$V = \frac{1}{2}m\omega^2x^2 \tag{37}$$

This potential is also of the form 2, so the propagator must have the form 20. This time, however, since $c \neq 0$, the function $A(t)$ will probably not have the form used in 21. The best we can say therefore is that

$$U\left(x,t;x',t'\right) = A(t)\, e^{iS_{cl}/\hbar} \tag{38}$$

where $A(t)$ has the form (from 19):

$$A(t) = \int_0^0 \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{m\dot{y}^2}{2} - \frac{1}{2}m\omega^2y^2\right) dt''\right]\mathfrak{D}\left[y\left(t''\right)\right] \tag{39}$$

We worked out the classical action for the harmonic oscillator earlier and found

$$S_{cl} = \frac{m\omega}{2\sin\omega t}\left[\left(x'^2+x^2\right)\cos\omega t - 2x'x\right] \tag{40}$$

where the particle is at x' at $t''=0$ and at x at $t''=t$. The propagator is therefore

$$U\left(x,t;x'\right) = A(t)\exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left(\left(x'^2+x^2\right)\cos\omega t - 2x'x\right)\right] \tag{41}$$

with $A(t)$ given by 39.

COMMENTS

Name: Alex

Error in equation 0.18 I believe the RHS should be equal to m/2, not m. I think this is also incorrect in Shankar.

Time: October 25, 2017 at 11:23 pm

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Fixed now. Thanks.

PINGBACKS

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