HARMONIC OSCILLATOR ENERGIES AND EIGENFUNCTIONS DERIVED FROM THE PROPAGATOR

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 8. Section 8.6, Exercise 8.6.3.

Given the propagator for the harmonic oscillator, it is possible to work backwards and deduce the eigenvalues and eigenfunctions of the Hamiltonian, although this isn't the easiest way to find them. We've seen that the propagator for the oscillator is

$$U\left(x,t;x'\right)=A\left(t\right)\exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left(\left(x'^{2}+x^{2}\right)\cos\omega t-2x'x\right)\right] \tag{1}$$

where $A\left(t\right)$ is some function of time which is found by doing a path integral. Shankar cheats a bit by just telling us what A is:

$$A\left(t\right)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}} \tag{2}$$

To deduce (some of) the energy levels, we can compare the propagator with its more traditional form

$$U\left(t\right)=\sum_{n}e^{-iE_{n}t/\hbar}\left|E_{n}\right\rangle \left\langle E_{n}\right| \tag{3}$$

where E_{n} is the nth energy level. In position space this is

$$U\left(t\right)=\sum_{n}\psi_{n}^{*}\left(x\right)\psi_{n}\left(x\right)e^{-iE_{n}t/\hbar} \tag{4}$$

We can try finding the energy levels as follows. We take $x=x'=t'=0$, which is equivalent to taking the end time t to be a multiple of a complete period of the oscillator, so that the particle has returned to its starting point. In that case, 1 becomes

$$U\left(x,t;x'\right)=A\left(t\right)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}} \tag{5}$$

If we can expand this quantity in powers of $e^{-i\omega t}$, we can compare it with the series 4 and read off the energies from the exponents in the series. To do this, we write

$$A(t) = \sqrt{\frac{m\omega}{\pi\hbar\left(e^{i\omega t}-e^{-i\omega t}\right)}} \tag{6}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-i\omega t/2}\frac{1}{\sqrt{1-e^{-2i\omega t}}} \tag{7}$$

To save writing, we'll define the symbol

$$\eta \equiv e^{-i\omega t} \tag{8}$$

so that

$$A(t) = \sqrt{\frac{m\omega}{\pi\hbar}}\eta^{1/2}\frac{1}{\sqrt{1-\eta^2}} \tag{9}$$

We can now expand the last factor using the binomial expansion to get

$$A(t) = \sqrt{\frac{m\omega}{\pi\hbar}}\eta^{1/2}\left[1+\frac{1}{2}\eta^2+\frac{3}{8}\eta^4+\ldots\right] \tag{10}$$

In terms of the original variables, we get

$$A(t) = \sqrt{\frac{m\omega}{\pi\hbar}}\left[e^{-i\omega t/2}+\frac{1}{2}e^{-5i\omega t/2}+\frac{3}{8}e^{-9i\omega t/2}+\ldots\right] \tag{11}$$

Comparing with 4, we find energy levels of

$$E = \frac{\hbar\omega}{2},\frac{5\hbar\omega}{2},\frac{9\hbar\omega}{2},\ldots \tag{12}$$

These correspond to $E_0, E_2, E_4, \ldots$. The odd energy levels $\left(\frac{3\hbar\omega}{2},\frac{7\hbar\omega}{2},\ldots\right)$ are missing because the corresponding wave functions $\psi_n(x)$ are odd functions of x and are therefore zero at $x=0$, so the corresponding terms in 4 vanish. The numerical coefficients in 11 give us $|\psi_n(0)|^2$ for $n = 0, 2, 4, \ldots$.

To get the other energies, as well as the eigenfunctions, from a comparison of 1 and 4 is possible, but quite messy, even for the lower energies. To do it, we take $t' = 0$ as before, but now we take $x = x' \neq 0$. That is, we start the oscillator off at some location $x' \neq 0$ and then look at it exactly one period later, when it has returned to the same position. The propagator 1 now becomes

$$U\left(x,t;x'\right)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left[\frac{im\omega}{2\hbar\sin\omega t}\left(2x^{2}\left(\cos\omega t-1\right)\right)\right] \tag{13}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar\left(e^{i\omega t}-e^{-i\omega t}\right)}}\exp\left[-\frac{m\omega}{\hbar\left(e^{i\omega t}-e^{-i\omega t}\right)}\left(x^{2}\left(\left(e^{i\omega t}+e^{-i\omega t}\right)-2\right)\right)\right] \tag{14}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\eta^{1/2}\frac{1}{\sqrt{1-\eta^{2}}}\exp\left[-\frac{m\omega x^{2}}{\hbar}\left(\frac{\frac{1}{\eta}+\eta-2}{\frac{1}{\eta}-\eta}\right)\right] \tag{15}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\eta^{1/2}\frac{1}{\sqrt{1-\eta^{2}}}\exp\left[-\frac{m\omega x^{2}}{\hbar}\left(\frac{1+\eta^{2}-2\eta}{1-\eta^{2}}\right)\right] \tag{16}$$

We now need to expand this in a power series in η, which gets very messy so is best handled with software like Maple. Shankar asks only for the first two terms in the series (the terms corresponding to $\eta^{1/2}$ and $\eta^{3/2}$) but even doing this by hand can get very tedious. The result from Maple is, for the first two terms:

$$\eta^{1/2}\rightarrow\sqrt{\frac{m\omega}{\pi\hbar}}e^{-m\omega x^{2}/\hbar}\eta^{1/2}=\sqrt{\frac{m\omega}{\pi\hbar}}e^{-m\omega x^{2}/\hbar}e^{-i\omega t/2} \tag{17}$$

$$\eta^{3/2}\rightarrow\sqrt{\frac{m\omega}{\pi\hbar}}\frac{2m\omega}{\hbar}e^{-m\omega x^{2}/\hbar}x^{2}\eta^{3/2}=\sqrt{\frac{m\omega}{\pi\hbar}}\frac{2m\omega}{\hbar}e^{-m\omega x^{2}/\hbar}x^{2}e^{-3i\omega t/2} \tag{18}$$

Comparing this with 4, we can read off:

$$E_{0}=\frac{\hbar\omega}{2} \tag{19}$$

$$\left|\psi_{0}\left(x\right)\right|^{2}=\sqrt{\frac{m\omega}{\pi\hbar}}e^{-m\omega x^{2}/\hbar} \tag{20}$$

$$E_{1}=\frac{3\hbar\omega}{2} \tag{21}$$

$$\left|\psi_{1}\left(x\right)\right|^{2}=\sqrt{\frac{m\omega}{\pi\hbar}}\frac{2m\omega}{\hbar}e^{-m\omega x^{2}/\hbar}x^{2} \tag{22}$$

To check this, we recall the eigenfunctions we worked out earlier, using Hermite polynomials

$$\psi_{n}(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^{n}n!}}H_{n}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^{2}/2\hbar} \tag{23}$$

The first two Hermite polynomials are

$$H_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = 1 \tag{24}$$

$$H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = 2\sqrt{\frac{m\omega}{\hbar}}x \tag{25}$$

Plugging these into 23 and comparing with 20 and 22 shows we got the right answer.